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WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. XXXIII.

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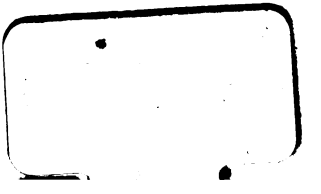

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Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR

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Page 60, line 26, *for* these *read* there.
Page 82, line 5 from bottom, *for* 2668 *read* 6268.

VOL. XXXII.

Page 72, *omit* from line 5 from bottom to line 8 from bottom of page 73; the paragraph to be left out belonging to the Solution of the closely related Question 5275, by the same author (Vol. XXXI., p. 27.)

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Solved Questions.

1433. (The Editor.)—Prove the following reciproca cases of involution:—

a. The three sides of every triangle, and every three concurrent lines through its three vertices, intersect every axis in six points in involution.

a'. The three vertices of every triangle, and every three collinear points on its three sides, subtend every vertex in six rays in involution.

b. The six perpendiculars on the six lines from any point in the former case determine at the point a pencil of six rays in involution.

b'. The six perpendiculars from the six points upon any line in the latter case determine on the line a system of six points in involution. 50

1588. (Professor Sylvester, F.R.S.)—Solve the equation

$$ax^5 + 5bx^4 + 10cax^3 + 10cbx^2 + 5c^2ax + c^2b = 0.$$ 115

1628. (Professor Sylvester, F.R.S.)—Extending the meaning of the word Bicorn to signify the general curve defined by the equations

$$x = \frac{c^3}{\phi^4 + \phi^3}, \quad y = \frac{c^2(2\phi + 3)}{\phi^3 + \phi^2},$$

show that the limiting form of the curve, as c converges towards zero, is a semicubical parabola and its axis extended indefinitely in both directions, forming together a sort of Trident. Explain the mode in which the geometrical passage of the double-horned to the tridentine form takes place, and specify in the Trident the morphological equivalents of the respective four branches of the general curve..... 79

2432. (Colonel Clarke, C.B., F.R.S.)—Two perfectly flexible and inextensible films, being equal circles in form, are fastened together along the perimeter; what is the greatest amount of air that this could be made to hold (as an *air-cushion*) ?..... 25

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2756. (J. Griffiths, M.A.)—Show that an infinite number of triangles can be described such that each has the same circumscribed, nine-point, and self-conjugate circles as a given triangle..... 45

2782. (Professor Crofton, F.R.S.)—(1) A given point is known to be within a certain circle of given radius, but unknown position; find the chance that another given point is also within the circle. Also (2) three given points are known to be within a certain circle, which is otherwise altogether unknown: determine the most probable position of its centre. Again, (3) two given points are known to be within a circle, and a third given point is known to be outside it: determine the most probable position of its centre. 18

3753. (W. H. H. Hudson, M.A.)—There are three towns A, B, C, forming a railway triangle; the population of each is proportional to the line joining the other two. Assuming that the number of trains in and out of each varies as its population, prove that the number of train-miles run between A and B varies as $\cos^2 \frac{1}{2}C$. Show also that, for a given aggregate population and length of line, the total number of train-miles is greatest when the towns are equidistant 26

3915. (J. J. Walker, M.A.)—From points on a diameter of a conic, tangents are drawn and perpendiculars let fall on the chords of contact; prove that the locus of the intersection of these last named lines is a hyperbola having as asymptotes the diameter conjugate to the given one and a perpendicular diameter. 76

3916. (S. Watson.)—If O, O_1, O_2, O_3 be the centres of the inscribed and escribed circles of a triangle ABC; show that the sum of the squares of the distances of the centroids of the triangles BOC, COA, AOB, BO_1C, CO_2A, AO_3B from the centroid of the triangle ABC, is $\frac{2}{9}(ab+bc+ca)+\frac{4}{9}D(D-r)$. 37

4031. (Professor Sylvester, F.R.S.)—Define *a tree* in its most general form regarded as a complex of relations between any given number of points or ideas..... 96

4143. (The late Professor Clifford, F.R.S.)—Three elastic strings without weight, whose natural lengths are OA, OB, OC, are joined together at O, the centre of the circumscribing circle of the horizontal triangle ABC; and a smooth sphere of given radius and weight is placed with its centre vertically above O, and allowed to descend until the centre rests at O. Find the moduli of elasticity in the three strings..... 18

4208. (Professor Monck, M.A.)—Three lines AD, BE, CF are drawn from the vertices of a triangle ABC through a point O to the opposite sides; prove that

$$2\Delta ABC : \Delta DEF = AO \cdot BO \cdot CO : DO \cdot EO \cdot FO$$..... 35

4549, 4566, 4578, 4579. (C. Leudesdorf, M.A., and E. B. Elliott, M.A.) A bag contains mn balls, each equally likely to be of any one of m different colours. All but m are drawn out at random, and found to be $n-1$ of each colour. Find the chance that the remaining m are (1) one of each colour, (2) all of one colour.... 68

4817. (Professor Sylvester, F.R.S.)—In a school of 15 girls, a rule has been laid down that they shall walk out every day in rows of threes, but that the same two girls shall never come together twice in the same row. The rule is supposed to have been carried out correctly during the six working days of the week,

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but when the time comes for their going to church together on Sunday it is found to be absolutely impossible to continue it any further. Investigate whether the rule *can* have been correctly carried out during the six previous days; and if so, show how. 53

5113. (The Editor.)—124, 154, 156, 163, 234, 132, 473, 456, 467, 367, and 124, 143, 126, 234, 167, 523, 173, 562, 567, 573 are two sets of ten triplets each made with seven elements, so that every duad employed occurs twice in either system. If we change them by any substitution, as 2135746—which means, put 2 for 1, 1 for 2, 3 for 3, 5 for 4, 7 for 5, 4 for 6, and 6 for 7—the first becomes 215, 275, 274, 243, 135, 231, 563, 574, 546, 346, which is similar to the first; but no substitution can make the first into the second, or the second into the first; so that these two are dissimilar sets. Find the entire number of dissimilar sets. 114

5138. (The Editor.)—Two persons make an appointment to meet on a certain day at the Medical Council Office, but without fixing the time further than that it is to be between 2 and 3 o'clock; and each agrees to wait ten minutes for the other. Now, supposing all times between the specified limits for coming to the Office to be equally probable, show, by a general solution, that the odds are 25 to 11 against the meeting taking place..... 78

5167. (Professor Seitz, M.A.)—A point is taken at random in the surface of a given circle, and from it a line equal in length to the radius is drawn, so as to lie wholly in the surface of the circle; find the chance that the line intersects a given diameter. 52

5264. (Rev. H. G. Day, M.A.)—Two similar polygons ABCDE..., $A_1B_1C_1D_1E_1$... are similarly situated on lines AB, A_1B_1 (not parallel); show that (1) if they are situated towards the same parts, a point O can be found at which AA_2, BB_1, CC_1, &c. all subtend the same angle: and (2) if towards opposite parts, a point O can be found at which the bisectors of the angles AOA_1, BOB_1, COC_1 coincide. 43

5272. (Professor Townsend, F.R.S.)—A solid of revolution of uniform density, turning without friction round a fixed point on its axis of figure, and rolling without sliding on a fixed plane inclined at any angle to the horizon, being supposed to have the velocity of rotation just sufficient to carry it with exhausted energy of motion to its position of unstable equilibrium against the action of gravity; determine, in finite terms, its time of passage from the opposite position of stable equilibrium to any other on its way..... 42

5329. (Professor Townsend, F.R.S.)—A rigid body, in unconstrained equilibrium in free space, being supposed set in motion by a single impulsive force applied at a definite point of its mass; if the initial motion be a pure rotation, show that—

(*a*) The several possible lines of impulsive action through the point generate a quadric cone, containing the three perpendiculars from the point on the three central principal planes, and the connector of the point with the centre of inertia of the body.

(*b*) The several corresponding axes of initial rotation in the body envelope a quadric cone, touching the three central principal planes, and the diametral plane conjugate to the direction of the point with respect to the central ellipsoid of the body.... 84

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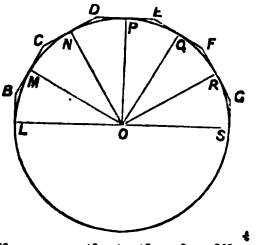
5355. (S. M. Drach, F.R.A.S.)—
Hindoo chess-inventors' claim bred despair;
'Twas doubling wheat-grains on following square.
How long 'twould feed mankind I here declare.
Four hundred millions' daily quartern loaf,
Ta'en from this fund, I find would be enough
From Noah's deluge human folk to sate,
Past one-nine-eighteen of the Christian date.
If useless Bacco-Bacchus men don't inhale,
Good corn-bread would suffice up to this tale. 46

5357. (Professor Sylvester, F.R.S.)—


In the annexed figure, O is the centre of the circle; LB, BC, CD, DE... are any number of tangents to the circle at L, M, N, P, ...; BC is bisected at M, CD at N, DE at P, ..., so that 2LB = BC = CD ... = 2GS. The lines touching the circle, and the radii drawn to the points of contact, are now supposed to be jointed rods moveable freely round each other at either extremity, and it is easily seen that the fan-like figure thus formed becomes a complete linkage, one (that is to say) capable of changing its form in only a single definite series of ways.

1. Prove that, when the fan is opened or closed, the alternate angles LOM, NOP, QOR, ... always remain equal to each other, as also the alternate angles MON, POQ, ROS, ..., so that the fan may be used to divide any angle into a given number of parts,—as, for example, in the figure above, when the fan is open or closed, the angle LON will always remain one-third of the angle LOS.

2. Imagine planes to become rigidly attached to BC, CD, DE, ..., and prove that, when OL is fixed, the trajectories of points taken in them are unicursal curves of the respective orders 97

5403. (Dr. Hopkinson, F.R.S.)—An iron hoop, 10 feet in diameter, revolves about an axis through its centre, perpendicular to its plane, with a velocity ω; find the greatest value ω can have without breaking the hoop. 53

5432. (Professor Ball, LL.D.)—If from any point perpendiculars are drawn to the generators of the surface $z(x^2+y^2)-2mxy=0$; show that the feet of the perpendiculars lie upon a plane ellipse. 86

5494. (Professor Tait.)—Determine the number of *distinct* knots with 8 crossings, and figure one form of each. 33

5507. (Elizabeth Blackwood.)—If A, B, C, D, E, F are random points on the surface of a sphere, find the probability that the plane through A, B, C will intersect the plane through D, E, F inside the sphere 87

5538. (Professor Townsend, F.R.S.)—A heavy flexible chain, of uniform thickness, being supposed to occupy longitudinally just half the interior of a slender circular tube, of uniform roughness, bounded radially by coaxal horizontal cylinders, and laterally by parallel vertical planes; determine, given all particulars, its two extreme positions of equilibrium under the action of gravity. 55

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5648. (J. L. McKenzie, B.A.)—P and Q are two given points, P on a given straight line, and Q on a given circle; show that it is possible to find another point R on the given circle, such that, if any straight line be drawn through R cutting the given circle in H and the given straight line in K, the points H, K, P, Q shall lie on a circle. 21

5706. (The Editor.)—Parallel to the base BC of a triangle ABC draw a straight line DE, cutting the sides AB, AC in D, E, such that the squares on BD and CE shall be together equal to the square on DE..... 89

5713. (Professor Sylvester, F.R.S.)—It is a well-known theorem of Euler's that the number of ways in which x can be formed by putting together j numbers restricted to the scale 0, 1, 2, ..., i is the coefficent of t^x in the quotient

$$\frac{(1-t^{i+1})(1-t^{i+2})\ldots(1-t^{i+j})}{(1-t)(1-t^2)\ldots(1-t^j)},$$

and consequently this quotient is Entire and Omnipositive.

Prove that the quotient obtained by writing $1-t^{i+kj}$, where k is any positive integer, in place of $1-t^{i+j}$, as the final factor in the numerator (leaving everything else unaltered), is also Entire and Omnipositive. 20

5745. (S. Tebay, B.A.)—A heavy egg-shaped shell, whose interior surface is perfectly smooth, is filled with incompressible fluid and placed upon a smooth horizontal plane. If it be twirled about a vertical axis through their common centre of gravity, determine the motion..... 37

5753. (W. R. Roberts, M.A.)—Show that the equation of the three tangents at the points of intersection of the straight line and cubic $L \equiv \lambda x+\mu y+\nu z$, $U \equiv x^3+y^3+z^3+6mxyz$, is $\Sigma U-(1+8m^3)\,L^2M = 0$, where Σ is the condition that L should touch U, and

$$M \equiv [\lambda^4-2\lambda(\mu^3+\nu^3)-6m\mu^2\nu^2]x+[\mu^4-2\mu(\lambda^3+\nu^3)+6m\lambda^2\nu^2]y+[\nu^4-2\nu(\lambda^3+\mu^3)-6m\lambda^2\mu^2]z. \quad \ldots\ldots 92$$

5758. (Rev. H. G. Day, M.A.)—If A, B be two fixed points, and P a variable point in the line AB, and if with A, P as foci an ellipse of eccentricity e_1 be drawn, and with B, P as foci an ellipse of eccentricity e_2, intersecting the former in Q; prove that the locus of Q is an ellipse with A, B as foci. 52

5762. (Professor Sylvester, F.R.S.)—If $f(x, y)$ is a quantic of n dimensions in x, y, prove that $(n-1)\left(\frac{df}{dx}\right)^2-nf\frac{d^2f}{dx^2}$ is a negative numerical multiplier of y^2 into the Hessian of f. 34, 92

5763. (Professor Sylvester, F.R.S.)—Prove that

$$\Sigma_{i=0}^{i=\infty}\frac{(-)^i x^{\frac{1}{2}i(i+1)}}{(1-x^k)(1-x^{k+1})\ldots(1-x^{k+i})},$$

when k is any positive integer, is a rational integral function of the degree k^2-k in x. Thus, for example,

$$\frac{1}{1-x}-\frac{x}{(1-x)(1-x^2)}+\frac{x^3}{(1-x)(1-x^2)(1-x^3)}\ldots\ldots = 1,$$

$$\frac{1}{1-x^2}-\frac{x}{(1-x^2)(1-x^3)}+\frac{x^3}{(1-x)(1-x^2)(1-x^3)}\ldots\ldots = 1-x+x^2. \quad 63$$

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5764. (Dr. Hirst, F.R.S.)—The projections, from a centre A, upon a fixed axis a, of n given lines $a_1, a_2 \ldots a_n$ in space, form a range which is homographic with a given range $A_1, A_2 \ldots A_n$. Prove that—(1) when $n = 4$, the locus of A is a quartic surface having a for a double line, and passing through each of the four given lines, as well as through each of the eight lines which can be drawn to intersect three of the latter and the axis a; (2) when $n = 5$, the locus of A is a curve of the seventh order, which cuts each of the five given lines four times, and the axis a six times; and (3) when $n = 6$, there are four positions of the centre A. 64

5782. (T. Mitcheson, B.A., L.C.P.)—If equilateral triangles be drawn on the sides of a triangle, prove that the lines that join the vertices of these triangles with the opposite corners of the original triangle, and the circles drawn round these triangles, meet at a point where the sides of the triangle subtend equal angles. ... 29

5815. (S. Tebay, B.A.)—Property worth £400 is mortgaged for s pounds, and afterwards sold out for an integral number (t) of pounds, which is s per cent. on the mortgage. If $s+t$ be $>$ £229, but $<$ £335, find the probable amount realised by the transaction. 47

5824 & 5903. (Professor Crofton, F.R.S.)—A convex figure whose perimeter is s and area Δ, is thrown at random on a fixed straight band of breadth a traced on a floor; show that the mean value of the area covered by it is $\frac{\pi a \Delta}{\pi a + s}$ or $\frac{\pi a r^2}{a + 2r}$, when the figure becomes a circle (omitting the cases when the figure does not meet the band). 95

5833. (Colonel Clarke, C.B., F.R.S.)—Two pairs of parallel lines cross a circle so as to form a rectangle within it; prove that (1) the chance that this rectangle contains a given point at the distance δ from the centre of the circle is $\frac{1}{4}(1-\delta^2)^2$; and (2) the chance that it contains a portion of a given diameter of the circle is $\frac{1}{4}(1+\log_e 4)$ 101

5845. (W. H. Lowry, M.A.)—Show that the equation whose roots are the cubes of the roots of $f(x) \equiv x^n + \mu x^{n-1} + \ldots = 0$, may be written $f(y^{\frac{1}{3}}) \{(\phi_1 y)^2 - \phi_2(y) \cdot \phi_3(y)\} = 0$;
and determine $\phi_1(y)$, $\phi_2(y)$, $\phi_3(y)$ 51

5868. (S. Tebay, M.A.)—Two vessels contain mixtures of wine and water; the first m gallons of wine, and a gallons of water; the second n gallons of wine, and b gallons of water; find the probability that a mixture formed at hazard will contain at least an average of the whole wine 51

5883. (Professor Tanner, M.A.)—Given k, a positive proper fraction, and the series $k_1, k'_2, \ldots k_1, k_2 \ldots$ formed from it by successive application of the equations $(1+k'_n)(1+k_{n-1}) = 2$, $k_n^2 + k'^2_n = 1$; show that, when n is indefinitely increased, $(1+k'_n)^{2^{n-1}}$ tends to the limit 1. 31

5909. (The Editor.)—A rod, capable of turning on a given point in its length, is placed at a fixed distance from a centre of force that varies inversely as the square of the distance; find its oscillations 103

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5914. (R. Tucker, M.A.)—How must a ball be struck from the edge of a circular billiard table, so that, after n reflexions, it may come back to the point of departure? Discuss the question for other positions of the ball 111

5917. (W. H. H. Hudson, M.A.)—If the capital that is invested in machinery and raw material be proportional to the rate of interest, and if the proceeds of labourers' work after replacing capital be proportional to their wages and to the aforesaid capital conjointly, and if the employers, after paying wages and interest on the aforesaid capital, make the greatest possible profit; prove that the rate of interest in the pound must be equal to the ratio of the proceeds above mentioned to twice the aforesaid capital..... 57

5932. (Professor Cochez.)—Connaissons les sommets de trois triangles équilatéraux construits sur les côtés d'un triangle, construire ce triangle. 30

5942. (C. Leudesdorf, M.A.)—Show that the equation of the segment of a circle which passes through (x_1, y_1), (x_2, y_2), and contains an angle θ, is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = \pm \cot\theta \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$. 38

5948. (A. Martin.)—A person throws 10 dice, and those that fall "ace up" are taken away. He throws the remainder, and those that fall "ace up" are taken away, and so on, till every die has been turned "ace up." Required the probability that this will be effected in 4 throws..... 101

5968. (Professor Benjamin Peirce, LL.D., F.R.S.)—If two bodies revolve about a centre, acted upon by a force proportional to the distance from the centre, and independent of the mass of the attracted body, prove that each will appear to the other to move in a plane, whatever be the mutual attraction. 91

5972. (Rev. W. A. Whitworth, M.A.)—Solve the equation

$$10x - x^3 = \sqrt{3} + \sqrt{2}.$$ 24

5977. (Edwyn Anthony, M.A.)—If θ, θ' be the eccentric angles corresponding to the extremities of the chord PQ of an ellipse, which is normal at P, $\tan\theta \tan\frac{1}{2}(\theta+\theta') = e^2-1$. 47

5982. (S. Tebay, B.A.)—Find three numbers such that, if the cube of their sum be increased by the product of every two of them, the three sums shall be cubes. 110

5991. (The Editor.)—A wire hoop is cut at random into three pieces; find the respective probabilities that the three pieces will, when straightened, admit of being formed into (1) a triangle of any kind, (2) an acute-angled triangle. 65

5992. (Elizabeth Blackwood.)—If P, Q, R, S be four random points on the surface of a given sphere; find the chance that the point S will be within the spherical triangle PQR. 54

5998. (C. Leudesdorf, M.A.)—A lamina in the form of an ellipse has one point P in its circumference fixed, and is struck perpendicularly to its plane at another point Q in the circumference; show that, if it begin to rotate about the tangent to the ellipse at P, then PQ is a diameter..... 46

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6009. (The late Professor Clifford, F.R.S.) — Find the form of a kite-string under the action of the wind. 59

6013. (Professor Minchin, M.A.)—If in any point of a body subject to stress the principal stresses consist of two tensions of intensities A and B (A > B) and a pressure of intensity C, show that the maximum intensity of shearing stress is $(AC)^{\frac{1}{2}}$, and find the plane on which it is exerted. If the principal stresses are a tension of intensity A and two pressures of intensities B and C (C > B), show that the maximum intensity of shearing stress is $(AC)^{\frac{1}{2}}$, and find the plane on which it is exerted..... 80

6031. (The Editor.)—Find the simplest possible six-valued function of six symbols 123456. 113

6043. (The Editor.)—Find the locus of the centres of circles tangential to a parabola and the tangent at its vertex. 40

6045. (R. F. Davis, M.A.)—APQ is a triangle similar to a given fixed triangle. If A be fixed, and the locus of P a straight line, show that the locus of Q is also a straight line..... 45

6047. (Donald McAlister, D.Sc.) — Prove the following rule for finding the centre of inertia of the surface of a right cone cut off by an oblique plane :—Make a plane section parallel to the base and at two-thirds of its distance from the vertex ; let the axis cut this plane in A, and let B be its centre ; then C, the point required, lies in AB produced, and AB : AC = 2 : 3..... 36

6048 & 6086. (J. Hammond, M.A.)—Prove that the n^{th} convergent to $2+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$ is $\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^{n}-(1-\sqrt{2},^{n}}$;

and to $2-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\ldots$ is $\frac{n+1}{n}$. 41

6060. (J. Young, B.A.)—Prove that any one of the three following equations is derivable from the other two :—

$$A \equiv a^2(x^2+xy+y^2) - axy(x+y) + x^2y^2 = 0,$$
$$B \equiv a^2(y^2+yz+z^2) - ayz(y+z) + y^2z^2 = 0,$$
$$C \equiv a^2(z^2+zx+x^2) - azx(z+x) + z^2x^2 = 0 \ldots\ldots 31$$

6061. (J. E. A. Steggall, M.A.)—If n letters are put at random into their n directed envelopes, find (1) the chance that no letter goes right ; and (2) show that when n is infinite the chance is e^{-1}. 57

6064. (Professor Townsend, M.A., F.R.S.)—A uniform cylindrical elastic rod, having its two extremities, its middle point, and the two points of bisection of its two semi-lengths, kept at a common horizontal level by rigid supports, being supposed in strained equilibrium, with slight deflection of form, under the action of gravity ; determine, by any method, the fraction of its weight divided equally between the supports at the two latter points. 24

6077. (W. J. C. Sharp, M.A.)—If any equation have three equal roots, prove that (1) the Hessian will have the same root as a double root ; or generally, (2) that if any equation have a p-ple root, the same quantity is a $2(p-1)$-ple root of the Hessian... 34

6097. (Professor Wolstenholme, M.A.)—Prove that

$$\int_0^\infty \frac{\log x}{(1+x)^4}\,dx = -\tfrac{1}{6}, \qquad \int_0^\infty \frac{1-3x}{(1+x)^5}(\log x)^2\,dx = 1,$$

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$$\int_0^\infty \frac{1-10x}{(1+x)^6}(\log x)^3 dx = -3, \quad \int_0^\infty \frac{1-25x+40x^2}{(1+x)^7}(\log x)^4 dx = 12, \text{ \&c.};$$

$$U_{n-2} = \int_0^\infty \frac{a_0 - a_1 x + a_2 x^2 - \dots}{(1+x)^{n+1}}(\log x)^{n-2} dx = (-)^n \frac{(n-2)!}{2}$$

if $a_0 = 1$, $a_1 = 2^{n-1} - (n+1)$, $a_2 = 3^{n-1} - (n+1)\,2^{n-1} + \frac{1}{2}(n+1)\,n$, and so on; the number of such coefficients being $\frac{1}{2}(n-1)$, or $\frac{1}{2}n$, as n is odd or even. 71

6098. (Professor Cochez.)—Trouver dans le plan de trois cercles un point tel que les tangentes menées de ce point aux troix cercles soient entre elles comme trois droites données. 55

6108 & 6109. (Elizabeth Blackwood and L. H. Rosenthal, B.A.)—P, Q, p, q are four points on a parabola, such that Pp and Qq are respectively parallel to the tangents at q and p. If O and O′ are the poles of pq and PQ respectively, and if O_{PQ} represents the perpendicular from O on PQ, &c., prove that (1) PQ = 3pq, (2) $O_{PQ} = O'_{pq}$, (3) $O'_{PQ} = 9O'_{pq}$, (4) $O_{PQ} = 5O_{pq}$; (5) find the ratio of the areas of the triangles Opq, O′PQ; (6) find the locus of the intersections with PQ of tangents at p, q, (α) when pq passes through the focus, and (β) when pq is parallel to a fixed line. 58

6111. (A. Johnson, M.A.)—A raft, whose length, breadth, and depth are l, b, d feet respectively, is made of wood, whose specific gravity is s; find the utmost weight it will bear before sinking. 44

6115. (A. J. C. Allen, B.A.)—TP, TQ are tangents at the ends of a focal chord PSQ of an ellipse, and the parallelogram TPRQ is completed; find the locus of R. 62

6116. (T. Morley, L.C.P.)—Solve the equations

$$x^3 - a = y^3 - b = z^3 - c = xyz. \quad \dots\dots 42$$

6119. (Professor Asaph Hall, M.A.)—If Δ_1, Δ_2, Δ_3, Δ_4 be the lengths of four parallel chords in an ellipse, and if (1 . 2), (1 . 3), (1 . 4), (2 . 3), (2 . 4), (3 . 4) denote the distances between these chords, prove that

$$+(2\,.\,3)\,(2\,.\,4)\,.\,(3\,.\,4)\,.\,\Delta_1^2 - (1\,.\,3)\,(1\,.\,4)\,(3\,.\,4)\,\Delta_2^2$$
$$+(1\,.\,2)\,(1\,.\,4)\,.\,(2\,.\,4)\,.\,\Delta_3^2 - (1\,.\,2)\,(1\,.\,3)\,(2\,.\,3)\,\Delta_4^2 = 0. \dots 27$$

6125. (Professor Monck, M.A.) — Find examples of right-angled triangles, with commensurable sides, such that (1) one, (2) neither of the acute angles is less than $\sin^{-1}\frac{3}{5}$. 56

6130. (The Editor.)—Let ABC be a triangle; D, E, F the points where its sides are touched by its inscribed circle; H, K, L the feet of the perpendiculars from the vertices of the triangle DEF on the opposite sides; Δ the area of the triangle ABC, and R, r the radii of its circumscribed and inscribed circles; Δ_1 the area of the triangle DEF, and Δ_2 the area of the pedal triangle HKL; then prove that Δ, Δ_1, Δ_2 are in continued proportion, whereof the common ratio is $\frac{r}{2R}$ 21

6132. (Elizabeth Blackwood.)—If O be the centre of a given circle of radius r, P and Q random points in it, and C the centroid of the triangle OPQ; show that the average value of OC is $\frac{128r}{135}$ ... 36

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6133. (H. McColl, B.A.)—It has been observed—(1) that, whenever the events A, B, X happen together, the events C, D, E also happen; (2) that, whenever the events B, C, Y happen together, the events D and E also happen; (3) that, whenever the event C or D happens or E fails to happen, then B or X happens or A fails to happen, and also C or Y happens or B fails to happen; (4) that, whenever X happens and A fails to happen, then Y happens and B fails to happen; and, conversely, whenever Y happens and B fails to happen, then X happens and A fails to happen. When may we conclude, from the occurrence or non-occurrence of events A, B, C, D, E (without any reference to Y)—(1) that the event X will happen (or has happened); (2) that the event X will *not* happen (or has not happened)? 22

6134. (T. Cotterill, M.A.)—Let aa', bb', cc' be the opposite vertices of four co-planar lines; then, if three systems of conics pass respectively through $(bb'cc')$, $(cc'aa')$, $(aa'bb')$; prove (1) that the invariant $\Theta\Theta' - \Delta\Delta'$ of two conics, one from each system, vanishes, and the double chord of intersection is harmonic to the third system of conics; and (2) give the reciprocal of these properties... 28

6136. (The Rev. W. A. Whitworth, M.A.)—Each of two urns contains n cards numbered successively 1, 2, 3, ... n. I draw a card from each urn and receive a number of shillings equal to the difference between the numbers on the two cards drawn. I repeat this operation, again receiving the difference in shillings between the two numbers which I draw; and so on, until at the n^{th} operation the cards are exhausted. Show that my expectation is worth $\frac{1}{3}(n^2-1)$ shillings..... 39

6142. (W. H. H. Hudson, M.A.)—Trace the curve

$$r(\cos 2\theta)^{\frac{1}{2}} = 2a\cos^2\theta,$$

and find its points of inflexion..... 39

6143. (C. B. S. Cavallin, M.A.)—Find the average area of the triangle cut off by a random line from an equilateral triangle.... 61

6146. (Rev. W. Roberts, M.A.)—Two tangents to a given parabola make angles with the axis such that the product of the tangents of their halves is constant; prove that the locus of the intersection of the tangents is a confocal parabola. 50

6147. (E. W. Symons, B.A.)—If A, B, C be the angles of a spheric triangle, and x, y, z any real quantities, prove that

$x^2+y^2+z^2-2yz\cos A-2zx\cos B-2xy\cos C$ is positive. 111

6153. (Sir James Cockle, F.R.S.)—When $y+z=v$ and $x+v=w$; then $v=0$ and $w=0$ each solve $(x+y)(1+p+q)+zp=0$...(1). Show, without recourse to the general primitive of (1), the distinct nature, and, by means of that primitive, the distinct origin of the two solutions; and that, taking $u=a$ and $v=b$ as independent particular integrals of (1), the Jacobian of u, v, w does not vanish identically. 19

6155. (Professor Cayley, F.R.S.)—Given by means of their metrical coordinates, any two lines; it is required to find their inclination, shortest distance, and the coordinates of the line of shortest distance..... 17

6156. (Professor Townsend, F.R.S.)—The axis of principal moment of any system of forces in a common space being supposed taken

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with respect to a variable point on any fixed right line in the space; show that it generates a skew paraboloid, whose axis of figure is the common perpendicular to the directive right line, and to the central axis of the system..... 33

6161. (Prof. Genese, M.A.)—If a straight line through the focus of a conic meet the tangents at the ends of the transverse axis in P, Q, prove that the circle whose diameter is PQ will touch the conic..... 116

6163. (Professor Matz, M.A.)—Prove that the mean distance of all the points of the cardioid-curve from the coordinate origin is $\frac{4}{3}a$. 83

6167. (Dr Hopkinson, F.R.S.)—Two magneto-electric machines with permanent inducing magnets, are used for the transmission of power, the first machine driven by an engine with uniform velocity, and producing a current which is passed through the second machine, the second machine acting as a magneto-electric engine. When the second machine is standing, but the current of the first is passed through its coil, the first machine is found to absorb a power W from the engine. Neglecting the loss due to friction, to disruptive discharges, and to short circuiting of the coils; prove (1) that the maximum power obtainable from the second machine is $\frac{1}{4}$W, and (2) that then the first machine absorbs $\frac{1}{8}$W from the engine. 32

6168. (The Editor.)—A coin is dropped at random over an indefinite horizontal grating formed of parallel equidistant wires; prove that, according as the distance between each pair of wires is equal to (1) the circumference, (2) the diameter, (3) the radius, of the coin, the respective chances that the coin will strike in passing through the grating are $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{\pi}$, $p_3 = \frac{1}{6\pi} + \frac{1}{4}\sqrt{3}$... 48

6169. (Elizabeth Blackwood.)—If P, Q, R are random points within a sphere of which O is the centre; find the average volume of the tetrahedron OPQR..... 75

6177. (B. Williamson, M.A.)—Suppose F, F_1, F_2 to be the three foci of a Cartesian oval, F_2 being the exterior focus, and let $FF_1 = c_2$, $FF_2 = c_1$, $F_1F_2 = c$; then, if $mr + lr_1 = nc_2$ be the equation of the inner oval referred to F and F_1, show (1) that its equation referred to F and F_2 is $nr + lr_2 = mc_1$; and referred to F_1 and F_2 is $mr_2 - nr_1 = lc$. Also prove (2) that the corresponding equations of the outer oval are obtained by changing l into $-l$ in the preceding..... 85

6178. (E. Anthony, M.A.)—Let D be the middle of the base BC of a triangle ABC; AL the perpendicular from A on BC; and AH, AH′ the bisectors of the angle A; then prove that

$$DL \cdot HH' = AB \cdot AC. \quad \text{..... 42}$$

6180. (R. E. Riley, B.A.)—From any point P of a parabola PM, PN are drawn perpendicular to the directrix and axis. If Q be a point in MN, dividing it in any constant ratio, prove that the locus of Q is a co-axal parabola. 44

6181. (Donald McAlister, D.Sc.)—Prove that the nth odd power of an odd number greater than unity can be presented as the difference of two square whole numbers in n different ways.... 71

6184. (G. Heppel, M.A.)—Find the number of terms in the expansion of the k^{th} power of the sum of n different quantities. 41

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6187. (A. Martin, M.A.)—Four cards are taken from a full pack; n other cards being drawn, none of them are aces; find the chance that the first four cards drawn were all aces. 84

6189. (Professor Johnson, M.A.)—A pack of cards may be shuffled in the following manner :—Take off with your right hand the top card of the pack (held in your left hand face upward), place the next card on top of it, the next below it, the next on top, and so on till the pack is exhausted. Then, if you take the 13 cards of one suit in order, and shuffle thus 10 times in succession, the cards are restored to their original order. On trial with other numbers of cards, it appeared that, in all cases, as soon as the original top card reappeared on the top of the pack, all the cards are in their original places. Prove this, and find the number of shuffles required for n cards. When an odd number of cards is used, the bottom card does not change its position; the above is therefore the result for 12 cards. But when the number of cards is odd, we obtain a new series of results by placing the second card below the first, the third above, the fourth below, and so on. In particular, 2^n cards or $2^n - 1$ cards require only $n+1$ shuffles. With the full pack the ninth shuffle brings all the aces together, all the deuces together, and so on; and three more shuffles (12 in all) restore the original order. A similar arrangement occurs when we use a pack formed of four equal packets. Thus, using 9 cards from each suit, this occurs on the sixth shuffle; three more shuffles in every case restoring the original order. The number of shuffles required are generally very small; a pack of fifty cards, however, requires 50 shuffles. 73

6190. (Professor Townsend, F.R.S.)—A uniform flexible cord, passing without friction through two fixed eyelets in its space, being supposed to hang between them in an arc of an equilateral hyperbola, and beyond them in the two intercepting radii of the curve, its two extremities just reaching the centre; show that the supposed position will be one of free equilibrium under the action of a uniform attractive force emanating from the centre of the curve. 90

6191. (Professor Crofton, F.R.S.)—Show that the common centre of gravity of any two arcs AB, A′B′ of the same circle, lies in the diameter perpendicular to the line joining the middles of the chords AA′, BB′..... 36

6192. (Professor Wolstenholme, M.A.)—Prove that any uniform lamina of mass nM is kinetically equivalent to a system of n equal particles, each of mass M, placed on the ellipse $Ax^2 + By^2 = 2AB$, at points whose successive eccentric angles differ by $\frac{2\pi}{n}$, the axes being the principal axes at the centroid, and nMA, nMB the principal moments of inertia..... 88

6198. (Dr. Hopkinson, F.R.S.)—At the Paris Exhibition a freezing machine (Giffard's) was exhibited, consisting of two cylinders, the pistons of which work on to two cranks on the same shaft, driven by an external source of power, and of a large air reservoir which is always maintained at the temperature of the external air. In the first cylinder air is compressed till its pressure is the same as in the reservoir, when valves open and

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the air passes, as the stroke is completed, into the reservoir. The second and smaller cylinder acts as an engine receiving compressed air from the reservoir for such a portion of the stroke that, being expanded for the remainder of the stroke, it is discharged at atmospheric pressure, but at a lower temperature. If V_1 and V_2 be the volumes of the cylinders, and Π the atmospheric pressure, and if the compression and expansion be supposed adiabatic, prove that the work done during each stroke in the first and second cylinders is

$$\Pi V_1 \frac{\gamma}{\gamma-1} \cdot \frac{V_1-V_2}{V_2}, \quad \Pi \frac{\gamma}{\gamma-1}(V_1-V_2). \ldots\ldots\ldots 97$$

6206. (H. McColl, B.A.)—Given (1) that, whenever the statements a, b, x are either all three true or all three false, then the statement c is false and y true, or else c is true and y false; (2) that, whenever d, e, y are either all three true or all three false, then the statement a is false and x true, or a is true and x false. When may we infer from these premises that either x or y is true? 60

6209. (C. Leudesdorf, M.A.)—A uniform inextensible string AB of length a is lying in a smooth horizontal plane in the form of the spiral whose intrinsic equation is $\log \frac{s}{c} = \frac{3\phi}{2}$, A being the point where $\phi = 0$. It is acted on by an impulsive tension at A; find the impulsive tension at any point P of the string; and show that, if P begin to move in a direction making an angle θ with the tangent at P to the spiral, then

$$\text{arc AP} = a\left(\frac{\tan\theta+2}{2-4\tan\theta}\right). \ldots\ldots\ldots 114$$

6210. (J. L. McKenzie, M.A.)—Prove that the conics of 5-point contact at any three collinear points on a cubic, meet the curve again in three collinear points. 90

6215. (Dr. Francesco Tirelli.)—Siano OA, OB, OC tre corde di un cerchio, sulle quali come diametri si descrivano tre circonferenze C_1, C_2, C_3. Indichiamo con A′, B′, C′ le ulteriori intersezioni di C_2 con C_3, C_3 con C_1, C_1 con C_2. Inoltre C_1 incontri OB ed OC in β e γ, C_2 incontri OC ed OA in γ_1 ed α_1 e C_3 incontri OA ed OB in α_2 e β_2. Dimostrare, che i punti A′, B′, C′ stanno per dritto, e che le rette $\beta\gamma$, $\gamma_1\alpha_1$, $\alpha_2\beta_2$ concorrono in uno stesso punto della A′B′C′..... 54

6223. (Professor Seitz, M.A.)—Within the perimeter ($= s$) of an n-gon circumscriptible about a circle of radius r, three points are taken at random; prove that (1) the chance that the circle through these three points will lie wholly within the polygon is $\frac{3}{8}\pi^2 r^2 s^{-2}$; and (2) when the polygon becomes a circle, the chance (which is then that of the Editor's Question 1843) is $\frac{3}{8}$. 115

6224. (Professor Morel.)—Eliminer ϕ et ψ de l'équation

$$Z = x\phi(z) + y\psi(z). \ldots\ldots\ldots 67$$

6229. (The Editor.)—Supposing the value of a rough diamond to vary as the square of its size; find how such a diamond would be reduced in value if it were broken at random into (1) two pieces, (2) three pieces..... 79

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6231. (R. Tucker, M.A.)—Eliminate θ and ϕ from the equations

$$\frac{x}{a}\cos\theta+\frac{y}{b}\cos\phi=\frac{x}{a}\sin\theta+\frac{y}{b}\sin\phi=1 \quad\ldots\ldots(1),$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\mu^2(\sin 2\theta-\sin 2\phi) \quad\ldots\ldots(2). \quad 76$$

6233. (C. Morgan, B.A.)—Pp, Pq are tangents to a parabola, and the bisectors of the angles pPq, PSp, PSq cut pq, Pp, Pq in f, K, K′ respectively; prove that Pf, KK′ intersect at right angles. 62

6234. (R. Rawson.)—When the roots of $x^3+nx^2+ax+b=0$ are all real; show, without the aid of Cardan's formula, that

$$4bn^3-a^2n^2-18abn+4a^3+27b^2<0,$$

and that one of the roots is less than $\frac{1}{6}\{2(n^2-3a)^{\frac{1}{2}}-n\}$. 89

6239. (T. Morley, L.C.P.)—The minor axis SA of an ellipse is equal to the distance between the foci; PSP′ is a focal chord of a parabola, whose focus is S and vertex A; AP, AP′ cut the ellipse in Q, Q′; prove that QSQ′ is a right angle. 62

6247. (Professor Wolstenholme, M.A.)—A point O is taken within a triangle ABC, and circles are drawn with centres A, B, C and radii OA sin BOC, OB sin COA, and OC sin AOB respectively; prove that the internal common tangents to these circles, taken two and two, will pass three of them through O, and the other three through another point O′, such that O, O′ are foci of an inscribed ellipse. [If O be taken outside, one pair of internal and two pair of external tangents must be taken.] ... 93

6252. (Dr. Hopkinson, F.R.S.)—It is found that a Leyden jar loses 0·000001 of its charge per second by conduction through the glass. The specific inductive capacity of the glass of which it is made is 8. Show that the resistance of a cubic centimetre of the glass is roughly 14×10^{17} ohms, having given that one electro-magnetic unit of quantity is equal to 3×10^{10} electrostatic units, and that one ohm is equal to 10^9 electro-magnetic units. 66

6253. (B. Williamson, F.R.S.)—If two straight lines in a moving plane area always touch the involutes to two circles, prove that any other straight line in the moving area will always touch the involute to a circle. 67

6258. (H. McColl, B.A.)—Assuming as an established law that the event C happens whenever A happens with X, and also whenever B happens without X; that C never happens when B happens with X, nor when A happens without X; that A happens on an average 3 times out of 5, and B once out of 3 times; that A, when it happens, is followed by C once in 10 times (on an average), and B by C 11 times in 20. Show that X will happen on an average between 63 and 83 times in 300. 113

6264. (R. E. Riley, B.A.)—If a circle be drawn with its centre at the vertex of the parabola $(x-y)^2=(32)^{\frac{1}{2}}a(x+y)$, and with a diameter equal to the latus-rectum, prove that the poles with respect to the parabola, of tangents to the circle, lie on the rectangular hyperbola $xy=2a^2$. 86

6267. (C. Leudesdorf, M.A.)—In a plane triangle R, r, r_1, r_2, r_3 are the radii of the circumscribed, inscribed, and three escribed

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circles; prove that, if
$(r_2+r_3-r_1)(r_3+r_1-r_2)(r_1+r_2-r_3) = -8r_1r_2r_3$,
then $r+4R$ will be equal to the perimeter of the triangle..... 83

6268. (N'Importe.)—A writes to B requiring an answer within n days. It is known that B will be at the address on some one of these days, any one equally likely. It is a p-days' post between A and B. If one in every q letters is lost in transit, find the chance that A receives an answer in time. $(n > 2p.)$..... 82

6275. (Professor Crofton, F.R.S.)—Show that the mean distance of the vertex of a triangle from all points in the area is equal to its distance from the centre of gravity, *measured along a parabolic path*, which leaves the vertex in the direction of one of the sides, and reaches the centre of gravity in a direction parallel to the other,—the axis of the parabola being parallel to the base..... 106

6282. (Professor Genese, M.A.)—ABCD is a quadrilateral, M, N are the middle points of the diagonals AHC, BHD; and the circles HAB, HCD meet in P, and the circles HBC, HDA in Q; prove that the points M, N, H, P, Q lie on one circle..... 99

6291. (R. Tucker, M.A.)—A certain hyperbola is the locus of the mid-points of chords of an ellipse, the perpendicular bisectors of which pass through a fixed point; find the envelop of this hyperbola when the fixed point lies on the auxiliary circle of the ellipse. 102

6296. (E. B. Elliott, M.A.)—A circular city has n gates, n straight streets along radii to these gates, and n circular streets concentric with its wall cutting these at right angles. A person starts from the centre along any one of the streets and turns whenever possible, but never inwards towards the centre. Show (1) that the whole number of possible routes by which he may thus leave the city is $n2^n$; and that the chance that the gate by which he leaves the city is the one in the direction towards which he originally started is, (2), if n is odd, 2^{1-n}; or (3), if n is even, $2^{1-n}+\frac{n!}{2^n(\frac{1}{2}n!)^2}$..... 100

6297. (C. Taylor, M.A.) — The tangent at P to a rectangular hyperbola meets a chord OQ at right angles in R; show that the diameters bisecting the chords PO and PQ bisect the angles between the diameters to P and R. 98

6299. (C. Leudesdorf, M.A.)—A circular hoop of radius a, lying on a smooth horizontal plane, is rotating with angular velocity ω about an axis through its centre perpendicular to its plane, and a particle is projected from the centre of the hoop with velocity v against its inner surface, supposed perfectly rough; show that, if e be the coefficient of elasticity, and n the ratio of the mass of the particle to that of the hoop, the particle will strike the hoop again after a time
$$\frac{2(2n+1)^2aev}{(2n+1)^2e^2v^2+(n+1)^2a^2\omega^2}.$$ 105

6300. (J. L. McKenzie, B.A.)—In any triangle that is not equilateral, if Δ be the area, and l, m, n be the lines bisecting the angles and terminated by the opposite sides, prove that
$$lmn > \frac{64\Delta^3}{(b+c)(c+a)(a+b)}$$ 112

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6302. (J. Young, B.A.) — If AB, AD be the greater sides of a quadrilateral ABCD inscribed in a circle, prove that

$$\frac{\sin(A+B)}{\sin(A+D)} = \frac{AB^2-CD^2}{AD^2-BC^2}$$ 105

6303. (D. Edwardes.)—If a quadrilateral, three of whose sides are fixed, be inscribable in a circle, prove that the line bisecting its diagonals touches a fixed parabola, with respect to which the fixed triangle is self-conjugate, and whose directrix passes through one of the vertices of the triangle..... 103

6304. (J. Hammond, M.A.) — If at any point of a parabola the normal be produced to meet the directrix in g, prove that the radius of curvature at P is double the length of Pg. 112

6318. (J. Venn, M.A.)—A, B, C are set to sort a heap of books; A taking the English political books, and the bound foreign novels; B the bound political works and the English novels, provided they are not political; and C the bound English works, and the unbound political novels. What works will be claimed by both A and B? Will any be claimed by all three? 99

6336. (Rev. H. G. Day, M.A.)—Through four given points draw lines which will form a quadrilateral similar to a given quadrilateral. 100

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6155. (By Professor CAYLEY, F.R.S.) — Given, by means of their metrical coordinates, any two lines; it is required to find their inclination, shortest distance, and the coordinates of the line of shortest distance.

N.B.—If λ, μ, ν are the inclinations of a line to three rectangular axes, and α, β, γ the coordinates to the same axes of a point on the line, then the metrical coordinates of the line are

$$a, \quad b, \quad c, \quad f, \quad g, \quad h,$$
$$= \cos\lambda, \ \cos\mu, \ \cos\nu, \ b\cos\nu - c\cos\mu, \ c\cos\lambda - a\cos\nu, \ a\cos\mu - b\cos\lambda,$$

satisfying identically the relations $a^2 + b^2 + c^2 = 1$, $af + bg + ch = 0$.

Solution by Professor GENESE, M.A.

1. Let θ be the angle between the given straight lines; then

$$\cos\theta = aa + bb' + cc', \text{ and } \sin^2\theta = L^2 + M^2 + N^2,$$

where $$L = \begin{vmatrix} b, & c \\ b', & c' \end{vmatrix}, \quad M = \begin{vmatrix} c, & a \\ c', & a' \end{vmatrix}, \quad N = \begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}.$$

2. The equations to the straight lines are

$$(bz - cy = f, \quad cx - az = g, \quad ay - bx = h) \ \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$(b'z - c'y = f', \quad c'x - a'z = g', \quad a'y - b'x = h') \ \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equations to a plane through (1), parallel to (2), will be found (using $af + bg + ch = 0$) to reduce to $Lx + My + Nz = -(a'f + b'g + c'h)$.....(3).

Therefore, the equation to plane through (2), parallel to (1), is

$$Lx + My + Nz = af' + bg' + ch' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The shortest distance between (1) and (2) is, therefore,

$$(af' + a'f + bg' + b'g + ch' + c'h) \div (L^2 + M^2 + N^2)^{\frac{1}{2}}.$$

3. Let A, B, C, F, G, H be the coordinates of this distance; then

$$A : B : C = L : M : N;$$

therefore $A = L \operatorname{cosec} \theta$, $B = M \operatorname{cosec} \theta$, $C = N \operatorname{cosec} \theta$.

Again, the condition that ABCFGH should intersect (1), is found by equating to zero the expression for their shortest distance; hence

$$aF + bG + cH + fA + gB + hC = 0,$$

and

$$a'F + b'G + c'H + f'A + g'B + h'C = 0;$$

also

$$AF + BG + CH = 0;$$

and from these equations we can obtain F, G, H.

Let

$$P \equiv \begin{vmatrix} f, & g, & h \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}, \qquad P' \equiv \begin{vmatrix} f', & g', & h' \\ a, & b, & c \\ a', & b', & c' \end{vmatrix};$$

then

$$F = \frac{P(a' \cos\theta - a) + P'(a \cos\theta - a')}{\sin^3\theta}, \text{ &c.}$$

2782. (By Professor Crofton, F.R.S.)—(1) A given point is known to be within a certain circle of given radius, but unknown position; find the chance that another given point is also within the circle. Also (2) three given points are known to be within a certain circle, which is otherwise altogether unknown: determine the most probable position of its centre. Again, (3) two given points are known to be within a circle, and a third given point is known to be outside it: determine the most probable position of its centre.

Solution by the Proposer.

The answer to (2) is, the centre of the minimum circle which satisfies the conditions, that is to say, the circumscribed circle of the triangle ABC, if the triangle is acute; if obtuse, then the circle on longest side as diameter: for, whatever be the radius of the circle, this point is a possible position for its centre; this is the case for no other point.

In (3) the centre is most likely to be at an infinite distance. If, however, the radius is not to exceed a certain limit, there must be a definite solution, though not easy to find.

[A Solution by Professor Wolstenholme is given on p. 28 of Vol. XII. of the *Reprint*.]

4143. (By the late Professor Clifford, F.R.S.)—Three elastic strings without weight, whose natural lengths are OA, OB, OC, are joined together at O, the centre of the circumscribing circle of the horizontal triangle ABC; and a smooth sphere of given radius and weight is placed with its centre vertically above O, and allowed to descend until the centre rests at O. Find the moduli of elasticity in the three strings.

Solution by A. MARTIN, M.A.; Prof. MATZ, M.A.; *and others.*

Let T_1, T_2, T_3 be the tensions of the strings fastened at A, B, C respectively; λ_1, λ_2, λ_3 their respective moduli of elasticity; W the weight, and r the radius of the sphere; and R the radius of the circumscribing circle of the triangle; then, we have

$$R(1+\lambda_1 T_1) = (R^2-r^2)^{\frac{1}{2}} + r\sin^{-1}\frac{r}{R};$$

therefore
$$\lambda_1 = \frac{(R^2-r^2)^{\frac{1}{2}} - R + r\sin^{-1}\frac{r}{R}}{RT} \quad \ldots\ldots(1).$$

For the equilibrium of the sphere, resolving forces vertically, we have

$$(T_1+T_2+T_3)\frac{r}{R} = W \quad \ldots\ldots(2).$$

Resolving forces perpendicular to AB, in the plane of ABC, we have

$$(T_2+T_3)\cos A = T_1\cos(B-C) \quad \ldots\ldots(3).$$

Eliminating T_2+T_3 from (2), (3), $T_1 = \frac{WR}{r}\cdot\frac{\cos A}{\cos A+\cos(B-C)}$ ...(4).

Substituting in (1), we obtain

$$\lambda_1 = r\left\{\frac{(R^2-r^2)^{\frac{1}{2}} - R + r\sin^{-1}\frac{r}{R}}{WR^2}\right\}\left\{1+\frac{\cos(B-C)}{\cos A}\right\}, \quad \lambda_2 = \&c., \lambda_3 = \&c.$$

6153. (By Sir JAMES COCKLE, F.R.S.)—When $y+z=v$ and $x+v=w$; then $v=0$ and $w=0$ each solve $(x+y)(1+p+q)+zp=0$.....(1). Show, without recourse to the general primitive of (1), the distinct nature, and, by means of that primitive, the distinct origin, of the two solutions; and that, taking $u=a$ and $v=b$ as independent particular integrals of (1), the Jacobian of u, v, w does not vanish identically.

Solution by the PROPOSER.

From $v=0$ we get $p=0$ and $1+q=0$, relations which reduce (1) to $0=0$, and show that $v=0$ is an integral capable of being completed into $v=b$. From $w=0$ we get $1+p=0$ and $1+q=0$, relations which reduce (1) to $-w=0$, not to $0=0$, and $w=0$ is a solution incapable of a similar completion. The integrating system of (1) is

$$dx : w = dy : (x+y) = -dz : (x+y) = d\mathrm{M} : (-2\mathrm{M}),$$

where M is JACOBI's multiplier. Hence $dw : w = -d\mathrm{M} : 2\mathrm{M}$ and $dv=0$; whence $v=b$, and a value of M is w^{-2} or $(x+b)^{-2}$. Integrating,

$$(x+b)^{-2}\{(x+y)\,dx-(x+b)\,dy\} = 0;$$

and, after the integration, replacing $x+b$ by w, we have

$$\frac{z}{w}+\log w = a, \text{ or, say, } u = a.$$

The general primitive of (1) is $u = \phi v$; and $w = 0$ gives $u^{-1} = 0$, since $w \log w$ vanishes with w. Putting $\frac{du}{dx} = \lambda$, the Jacobian of u, v, w is

$$\begin{vmatrix} \lambda, & \lambda, & \lambda + \frac{1}{w} \\ 0, & 1, & 1 \\ 1, & 1, & 1 \end{vmatrix}, = -\frac{1}{w},$$

and $= \infty$ when $w = 0$.

But, according to Boole (implicitly at pp. 331, 332 of his *Differential Equations*, 1865, and explicitly at pp. 60, 61 of the *Supplement*), this determinant ought to vanish identically.

5713. (By Professor Sylvester, F.R.S.)—It is a well-known theorem of Euler's that the number of ways in which x can be formed by putting together j numbers restricted to the scale 0, 1, 2, ..., i is the coefficient of t^x in the Quotient $\frac{(1-t^{i+1})(1-t^{i+2}) \ldots\ldots (1-t^{i+j})}{(1-t)(1-t^2) \ldots\ldots (1-t^j)}$,

and consequently this Quotient is Entire and Omnipositive.

Prove that the quotient obtained by writing $1-t^{i+kj}$, where k is any positive integer, in place of $1-t^{i+j}$, as the final factor in the numerator (leaving everything else unaltered), is also Entire and Omnipositive.

[Prof. Sylvester remarks that this is not a *fancy* question, but one that presented itself in a very important research in the Theory of Invariants.]

Solution by W. J. Curran Sharp, M.A.

If $\frac{(1-t^{i+1})(1-t^{i+2}) \ldots\ldots (1-t^{i+j})}{(1-t)(1-t^2) \ldots\ldots (1-t^j)}$ be denoted by u_j,

$$\frac{(1-t^{i+1})(1-t^{i+2}) \ldots\ldots (1-t^{i+j-1})(1-t^{i+kj})}{(1-t)(1-t^2) \ldots\ldots (1-t^{j-1})(1-t^j)}$$

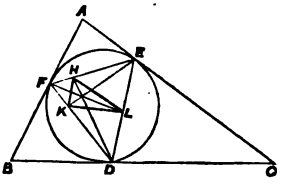
$$= u_{j-1} \frac{1-t^{i+kj}}{1-t^j} = u_{j-1} \frac{1-t^{i+j}+t^{i+j}-t^{i+kj}}{1-t^j} = u_j + u_{j-1} t^{i+j} \frac{1-t^{j(k-1)}}{1-t},$$

which proves what is required, as u_j, u_{j-1}, and $\frac{1-t^{j(k-1)}}{1-t^j}$ are all three Entire and Omnipositive.

6130. (By the Editor.)—Let ABC be a triangle; D, E, F the points where its sides are touched by its inscribed circle; H, K, L the feet of the perpendiculars from the vertices of the triangle DEF on the opposite sides; Δ the area of the triangle ABC, and R, r the radii of its circum-

scribed and inscribed circles; Δ_1 the area of the triangle DEF, and Δ_2 the area of the pedal triangle HKL; then prove that Δ, Δ_1, Δ_2 are in continued proportion, whereof the common ratio is $\frac{r}{2R}$.

I. *Solution by* C. MORGAN, B.A.; Prof. GENESE, M.A.; *and others.*



The angles of the triangle DEF, and of its pedal triangle HKL, are, obviously,

$$\angle D = \tfrac{1}{2}(\pi - A), \text{ \&c. \&c.;}$$
$$\angle H = \pi - 2D = A, \text{ \&c. \&c.;}$$

also, the radius of the circle around HKL, which is the nine-point circle of the triangle DEF, is $\frac{1}{2}r$. Thus, since the triangles ABC, HKL are *similar*, and have their *linear* dimensions proportional to the radii of their circumscribed circles, we have $\Delta^{\frac{1}{2}} : \Delta_2^{\frac{1}{2}} = R : \frac{1}{2}r = 2R : r$.

[The sides of the triangle DEF are plainly parallel to the sides of its escribed triangle $O_1O_2O_3$; also ABC is the pedal triangle of the triangle $O_1O_2O_3$; hence the triangle ABC bears the same relation to the triangle $O_1O_2O_3$ as the triangle HKL does to the similar triangle DEF; hence we have

$$\Delta : \Delta_2 = \text{triangle } O_1O_2O_3 : \text{triangle DEF}$$
$$= \text{circle } O_1O_2O_3 : \text{circle DEF} = (2R)^2 : r^2.$$

We thus see at once that the triangles HKL, DEF, ABC, $O_1O_2O_3$ are in continued proportion, the common ratio being as stated in the Question.]

II. *Solution by* Prof. COCHEZ; A. MCMURCHY, B.A.; *and others.*

The triangle HKL is similar to ABC; hence, we have

$$BC = 2R \sin A, \qquad EF = 2r \sin D = 2r \cos \tfrac{1}{2}A,$$
$$KL = EF \cos D = EF \sin \tfrac{1}{2}A = r \sin A;$$

therefore we have $\Delta^{\frac{1}{2}} : \Delta_2^{\frac{1}{2}} = 2R : r$.

Again, $\Delta_1 = \frac{1}{2}DE \cdot DF \cdot \sin D = 2r^2 \sin D \sin E \sin F$

$$= 2r^2 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C = \frac{2r^2 s \Delta}{abc} = \frac{r}{2R} \cdot \Delta;$$

hence Δ, Δ_1, Δ_2 are in the continued proportion specified.

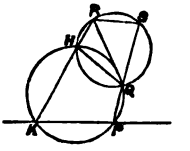
5648. (By J. L. MCKENZIE, B.A.)—P and Q are two given points, P on a given straight line, and Q on a given circle; show that it is possible to find another point R on the given circle, such that, if any straight line be drawn through R cutting the given circle in H and the given straight line in K, the points H, K, P, Q shall lie on a circle.

Solution by the EDITOR.

Let PQ meet the given circle in a second point S; then the points H, K, P, Q will lie on a circle, provided that the angle KPQ = QHR,

or $\angle KPQ + RSQ = QHR + RSQ = 180°$;

whence SR is parallel to PK, and the construction is obvious.



6133. (By H. McCOLL, B.A.)—It has been observed (1) that, whenever the events A, B, X happen together, the events C, D, E also happen; (2) that whenever the events B, C, Y happen together, the events D and E also happen; (3) that whenever the event C or D happens or E fails to happen, then B or X happens or A fails to happen, and also C or Y happens or B fails to happen; (4) that whenever X happens and A fails to happen, then Y happens and B fails to happen; and, conversely, whenever Y happens and B fails to happen, then X happens and A fails to happen. When may we conclude, from the occurrence or non-occurrence of the events A, B, C, D, E (without any reference to Y)—(1) that the event X will happen (or has happened); (2) that the event X will *not* happen (or has not happened)?

I. *Solution by the* PROPOSER.

The premises of the problem are as follows:—

$abx : cde,\ boy : de,\ c+d+e' : (a'+b+x)(b'+c+y),\ a'x = b'y \ldots (1, 2, 3, 4).$

From these premises, by rule 23 and formula 6 of my third paper in the *Proceedings* of the London Mathematical Society, we get

$$x'y : (b'+c'+de)(c'd'e+a'+b)\,b, \quad x'y' : c'd'e+(a'+b)(b'+c);$$

that is, $x'y : (c'+de)\,b, \quad x'y' : c'd'e+a'b'+a'c+bc,$

for, from formula 2, $(c'd'e+a'+b)\,b = (c'd'e+a'+1)\,b = b.$

Hence, adding antecedent to antecedent, and consequent to consequent, we have $x' : b+bde+c'd'e+a'b'+a'c,$

for $x'(y+y') = x', \quad \text{and} \quad b(c+c') = b.$

Omitting the redundant term bde, and observing that $b+a'b' = b+a'$, this becomes $x' : b+a'+a'c+c'd'e.$

Omitting the redundant term $a'c$, and transposing, we have

$$ab'(c+d+e') : x.$$

Translated into ordinary language, this may be read thus:—When we know that the event A has happened, and that B has not happened, and also that either C or D has happened or that E has not happened, then we may infer that X will happen or has happened.

Again, from the premises of the problem, we get

$xy : (a'+b'+cde)(b'+c'+de)(ab+a'b'), \quad xy' : (a'+b'+cde)(c'd'e+b'+c)\,a.$

Using formula 5 of the before-cited paper, and omitting the term a' in the first bracket of the second consequent, because it is inconsistent with the last factor a, these two implications become

$$xy : (b' + a'c' + a'de + cde)(ab + a'b'), \quad xy' : (b' + cde)\,a.$$

The term $a'de$ in the first bracket of the first consequent is redundant (see rule 20), for, by rule 22,

$$a'de\,(b' + a'c' + cde)' = a'de\,(b' + c' + e)' = 0.$$

Omitting this redundant term, then multiplying the factors, and adding antecedent to antecedent, and consequent to consequent, we get

$$x : ab' + acde + \underline{abcde} + a'b'.$$

Omitting the redundant term underlined, and observing that

$$ab' + a'b' = (a + a')\,b' = b',$$

this becomes $x : b' + acde$, or, transposing, $b\,(a' + c' + d' + e') : x'$.

Translated into ordinary language, this may be read as follows:—When we know that the event B has happened, and that one (or more) of the events A, C, D, E has *not* happened, then we may infer that the event X will not happen, or has not happened.

[Professor JEVONS's method of solving logical problems—adopted by Professor TANNER in his solution of Quest. 5861 (*Reprint*, Vol. XXXI., p. 43)—is simple in theory, and when (as in Quest. 5861) only 4 or 5 elementary constituents have to be taken into account, it is not very tedious in practice; but the difficulty of applying his method—as he frankly points out himself in his *Elementary Lessons in Logic*—increases rapidly with the number of elementary constituents, and when these are 7 or 8 in number the method is thought to become altogether impracticable.]

II. *Solution by* C. J. MONRO, M.A.

Put H for cde, and K for what Mr. McCOLL writes $c + d + e'$; then, in BOOLE's notation, only with accents to negative symbols,

$$abH'x = 0, \quad b\,(c - H)\,y = 0, \quad K\,(ab'x' + bc'y') = 0,$$
$$a'x\,(1 - b'y) + (1 - a'x)\,b'y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1\text{—}4).$$

The left sides consist of positive class terms, and may be combined by addition.

Add (2), (3), (4), making y successively 1 and 0. This gives

$$b\,(c - H) + Kab'x' + a'bx + (1 - a'x)\,b' \quad \text{and} \quad Kbc' + Kab'x' + a'x.$$

The product of these gives the result of eliminating y, which, therefore (since $c'H = 0$), is $\quad a'b\,(c + H')\,x + Kab'x' = 0.$

Add (1); the whole system, irrespectively of y, is then reduced to

$$(a'bc + bH')\,x + Kab'x' = 0 \ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

Hence $\quad x = \dfrac{Kab'}{Kab' - a'bc - bH'}, \quad = \dfrac{0}{-H'}\,ab + \dfrac{K}{K}\,ab' + \dfrac{0}{-c - H'}\,a'b + \dfrac{0}{0}\,a'b';$

which, since $c + H' = 1 + e\,(1 - de)$ and cannot vanish,

$$= \tfrac{0}{0}\,abH + 0\,abH' + ab'K + \tfrac{0}{0}\,ab'K' + 0\,a'b + \tfrac{0}{0}\,a'b';$$

that is to say (since $K' = c'd'e$), that, irrespectively of Y, X always happens when A happens without B, but with C or with D or without E

inexclusively, as is evident from (3); and sometimes happens, and sometimes not, when all the other events happen, or A and E without the rest, or neither A nor B; but never when B happens without A, as is evident from (4), or with A unless with all the rest, as is evident from (1); or, in other words (as I see on reading the PROPOSER's solution), when B happens but at least one of the others does not.

5972. (By the Rev. W. A. WHITWORTH, M.A.)—Solve the equation

$$10x - x^3 = \sqrt{3} + \sqrt{2}.$$

Solution by Rev. Dr. KNISELY; Rev. J. L. KITCHIN, M.A.; *and others.*

The equation may be put into the following successive forms:—

$$x^4 - 10x^2 = -(\sqrt{3} + \sqrt{2})\,x,$$

$$x^4 - (5 + 2\sqrt{6})\,x^2 = (\sqrt{3} - \sqrt{2})^2 x^2 - (\sqrt{3} + \sqrt{2})\,x,$$

$$\{x^2 - \tfrac{1}{2}(5 + 2\sqrt{6})\}^2 = (\sqrt{3} - \sqrt{2})^2 x^2 - (\sqrt{3} + \sqrt{2})\,x + \tfrac{1}{4}(5 + 2\sqrt{6})^2;$$

whence, observing that $5 + 2\sqrt{6} = (\sqrt{3} + \sqrt{2})^2 = \dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$,

so that the right-hand side is a complete square, we have

$$x^2 - \tfrac{1}{2}(5 + 2\sqrt{6}) = \pm\,[(\sqrt{3} - \sqrt{2})\,x - \tfrac{1}{2}(5 + 2\sqrt{6})];$$

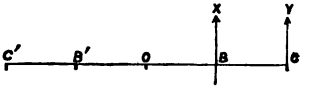
therefore $x = \sqrt{3} - \sqrt{2}$, or $\tfrac{1}{2}[-\sqrt{3} + \sqrt{2} \pm (25 + 6\sqrt{6})^{\frac{1}{2}}]$.

6064. (By Professor TOWNSEND, M.A., F.R.S.)—A uniform cylindrical elastic rod, having its two extremities, its middle point, and the two points of bisection of its two semi-lengths, kept at a common horizontal level by rigid supports, being supposed in strained equilibrium, with slight deflection of form, under the action of gravity; determine, by any method, the fraction of its weight divided equally between the supports at the two latter points.

I. *Solution by* T. R. TERRY, M.A.; A. MCMURCHY, B.A.; *and others.*

Let E be the coefficient of elasticity, and I the moment of inertia of a vertical section of the rod about a horizontal line drawn in the plane of section through the neutral axis.

Let $4a$ be the length of the rod, and $4aw$ its weight, X and Y being the pressures at B and C.



For segment CB, take C as origin and CB as axis of x, then $\mathrm{EI}\dfrac{d^2y}{dx^2} = \mathrm{Y}x - \tfrac{1}{2}wx^2$;

but $y = 0$ when $x = 0$ and when $x = a$, hence we have

$$\mathrm{EI}y = \tfrac{1}{6}\mathrm{Y}(x^3 - a^2x) - \tfrac{1}{24}w(x^4 - a^3x);$$

and, at B, we have $\mathrm{EI}\dfrac{dy}{dx} = \tfrac{1}{3}\mathrm{Y}a^2 - \tfrac{1}{8}wa^3$ ..... (1).

For the segment BO, take B as origin and BO as axis of x, then we have

$$\mathrm{EI}\frac{d^2y}{dx^2} = \mathrm{X}x + \mathrm{Y}(x+a) - \tfrac{1}{2}w(x+a)^2;$$

but $y = 0$ when $x = 0$ and when $x = a$, hence we have

$\mathrm{EI}y = \tfrac{1}{6}\mathrm{X}(x^3 - a^2x) + \tfrac{1}{6}\mathrm{Y}(x^3 + 3ax^2 - 4a^2x) - \tfrac{1}{24}w(x^4 + 4a^3x + 6a^2x^2 - 11a^3x)$;
therefore, at B and O, respectively, we have

$$\mathrm{EI}\frac{dy}{dx} = -\frac{a^2}{6}\mathrm{X} - \frac{4a^2}{6}\mathrm{Y} + \tfrac{11}{24}a^3w \text{ .....(2),}$$

$$\mathrm{EI}\frac{dy}{dx} = \frac{a^2}{3}\mathrm{X} + \frac{5a^2}{6}\mathrm{Y} - \tfrac{7}{24}a^3w \text{ ....(3).}$$

But expressions (1) and (2) must be equal, and by symmetry we see that $\dfrac{dy}{dx}$ vanishes at the point O; hence we obtain, finally,

$$4\mathrm{X} + 24\mathrm{Y} = 14aw, \quad 8\mathrm{X} + 20\mathrm{Y} = 17aw,$$

$$\mathrm{X} = \frac{8aw}{7} = \tfrac{8}{7} \text{ weight of the rod.}$$

II. *Solution by the* PROPOSER.

Denoting by a the semi-length of the rod, by W its semi-weight, and by W′ the fraction of W supported by each prop in question, and supposing the latter removed and replaced each by a force equal and opposite to W′. Then, since, on known principles, the depression y, at any point of the rod distant x from its centre, due to the action of gravity, is given by the formula

$$\mathrm{EI}\,.\,y = \tfrac{1}{48}\,.\,\frac{\mathrm{W}}{a}\,.\,x\,.\,(a-x)\,.\,(2a^2 - 3ax + 2x^2);$$

and the elevation y', at any point of it, distant x' from its centre, due to the action of the upward force W′, by the formula

$$\mathrm{EI}\,.\,y' = \tfrac{1}{12}\,.\,\frac{\mathrm{W}}{a^3}\,.\,x'^2\,.\,(a-x')^2\,.\,(a^2 + 4ax' - x'^2),$$

where E and I have their ordinary well-known significations; hence, in order that y' should neutralise y at the point for which $x' = x = \frac{1}{2}a$, we must have $\tfrac{11}{96}\mathrm{W}' = \tfrac{1}{24}\mathrm{W}$, or $\mathrm{W}' = \tfrac{4}{11}\mathrm{W}$,
which accordingly is the fraction required.

2432. (Proposed by Colonel CLARKE, C.B., F.R.S.)—Two perfectly flexible and inextensible films, being equal circles in form, are fastened together along the perimeter; what is the greatest amount of air that this could be made to hold (as an *air-cushion*)?

Remarks by J. J. WALKER, M.A.

Supposing two such films could form the surfaces bounding two solids, connected along the perimeters of the given circles, and convex outwardly from symmetry, it is apparent that either part would be a surface of revolution round the line joining the centres as axis; and the question would be reduced to "finding the curve of given length such that the surface generated by its revolution round a given line may be given, and that the volume of the generated solid may be a maximum." These conditions are too many. For the solutions of the problem resulting from omitting either of the first two conditions, reference may be made to Professor JELLETT'S *Calculus of Variations*, pp. 160, 161. It does not seem possible to form a cushion having a continuous surface from two such films, connected as in the question, unless a *corrugated* shape were admissible and practicable.

[Col. CLARKE states that the problem is to determine the law of the corrugations or creasings (infinitely small); that he believes the law in question to be simple, and that he once investigated it, and, in illustration thereof, folded some paper, of which he sends a specimen.]

3753. (By W. H. H. HUDSON, M.A.)—There are three towns A, B, C, forming a railway triangle; the population of each is proportional to the line joining the other two. Assuming that the number of trains in and out of each varies as its population, prove that the number of train-miles run between A and B varies as $\cos^2\frac{1}{2}C$. Show also that, for a given aggregate population and length of line, the total number of train-miles is greatest when the towns are equidistant.

I. *Solution by* W. SIVERLY; J. A. KEALY, M.A.; *and others.*

Let x, y, z be the number of trains between (A, B), (A, C), (B, C), and λ some constant; then we have $x+y=\lambda a$, $x+z=\lambda b$, $y+z=\lambda c$. Putting, as usual, $\frac{1}{2}(a+b+c)=s$, and $s-a=s_1$, &c., we have $x=\lambda s_1$, $y=\lambda s_2$, $z=\lambda s_3$.

Assume $\lambda=\frac{s}{abc}\lambda'$, then $x=\frac{\lambda' ss_3}{abc}$, $y=\frac{\lambda' ss_2}{abc}$, $z=\frac{\lambda' ss_1}{abc}$;

hence the train-miles between A and B $=\frac{\lambda_1 ss_3}{ab}$ varies as $\cos^2\frac{1}{2}C$, between A and C varies as $\cos^2\frac{1}{2}B$, and between B and C as $\cos^2\frac{1}{2}A$. Also $A+B+C=\pi$, $u=\cos^2\frac{1}{2}A+\cos^2\frac{1}{2}B+\cos^2\frac{1}{2}C=\max$. Differentiating with respect to A and B successively, we have

$$1+\frac{dC}{dA}=0,\quad 1+\frac{dC}{dB}=0;$$

$$\frac{du}{dA}=-\tfrac{1}{2}\sin A-\tfrac{1}{2}\sin C\frac{dC}{dA},\quad \frac{du}{dB}=-\tfrac{1}{2}\sin B-\tfrac{1}{2}\sin C\frac{dC}{dB};$$

whence, replacing $\frac{dC}{dA}$, $\frac{dC}{dB}$ by their values, we obtain

$$\tfrac{1}{2}\sin A=\tfrac{1}{2}\sin B=\tfrac{1}{2}\sin C,\text{ therefore } A=B=C=\tfrac{1}{3}\pi.$$

II. *Solution by the* PROPOSER.

Let A_b be the number of trains from A to B, &c.; then we have

$$A_b + A_c = B_a + C_a, \quad B_c + B_a = C_b + A_b, \quad C_a + C_b = A_c + A_b;$$

which are only two independent equations. Also the Question gives

$$\frac{A_b + A_c}{a} = \frac{B_c + B_a}{b} = \frac{C_a + C_b}{c} = \lambda, \text{ suppose; therefore } \lambda = \frac{A_b + B_a}{a+b-c};$$

therefore the number of train-miles between A and B is

$$c(A_b + B_a) = c\lambda(a+b-c)\, a \cos^2 \tfrac{1}{2}C;$$

hence the total number of train-miles is

$$\lambda \left\{ a(b+c-a) + b(c+a-b) + c(a+b-c) \right\}$$

$$= \frac{\lambda}{3} \left\{ (a+b+c)^2 - 2(a-b)^2 - 2(b-c)^2 - 2(c-a)^2 \right\};$$

and since $a+b+c$ is given, this is greatest when $a = b = c$.

6119. (By Professor ASAPH HALL, M.A.)—If Δ_1, Δ_2, Δ_3, Δ_4 be the lengths of four parallel chords in an ellipse, and if (1 . 2), (1 . 3), (1 . 4), (2 . 3), (2 . 4), (3 . 4) denote the distances between those chords, prove that

$$+(2.3)(2.4).(3.4).\Delta_1^2 - (1.3)(1.4)(3.4)\Delta_2^2$$
$$+(1.2)(1.4).(2.4).\Delta_3^2 - (1.2)(1.3)(2.3)\Delta_4^2 = 0.$$

I. *Solution by* Prof. GENESE, M.A.; T. R. TERRY, M.A.; *and others.*

Since the distance between two parallel chords is proportional to the part of the conjugate diameter intercepted by them, the property in question is clearly projective. In the circle of unit radius, if Δ_1 subtend an angle 2α at the centre, the first term becomes

$$(\cos\beta - \cos\gamma)(\cos\beta - \cos\delta)(\cos\gamma - \cos\delta)(1 - \cos^2\alpha)$$
$$= (\cos^2\beta\cos\gamma - \cos\beta\cos^2\gamma + \cos^2\gamma\cos\delta - \cos\gamma\cos^2\delta$$
$$+ \cos\beta\cos^2\delta - \cos^2\beta\cos\delta)(1 - \cos^2\alpha).$$

If to this we add the three symmetrical terms, the result is zero.

[Prof. HALL states that this theorem was announced by SAVARY in his Memoir on the Orbits of Comets, in the *Conn. des Temps* for 1830, p. 65.]

II. *Solution by* Prof. WOLSTENHOLME, M.A.; Prof. EVANS, M.A.; *and others.*

Refer the ellipse to a diameter bisecting the chords, and any line parallel to the chords, then its equation will be of the form

$$4Ay^2 + Bx^2 + Cx + D = 0;$$

and if x_1, x_2, x_3, x_4 be the values of x for the four chords, we shall have the four equations,

$$A\Delta_1^2 + Bx_1^2 + Cx_1 + D = 0, \text{ \&c.} = 0, \text{ \&c.} = 0, \text{ \&c.} = 0; \text{ whence}$$

$$\begin{vmatrix} \Delta_1^2, & x_1^2, & x_1, & 1 \\ \Delta_2^2, & x_2^2, & x_2, & 1 \\ \Delta_3^2, & x_3^2, & x_3, & 1 \\ \Delta_4^2, & x_4^2, & x_4, & 1 \end{vmatrix} = 0, \text{ or } \Delta_1^2 \begin{vmatrix} x_2^2, & x_2, & 1 \\ x_3^2, & x_3, & 1 \\ x_4^2, & x_4, & 1 \end{vmatrix} - \Delta_2^2 \begin{vmatrix} x_1^2, & x_1, & 1 \\ x_3^2, & x_3, & 1 \\ x_4^2, & x_4, & 1 \end{vmatrix} + \text{\&c.} = 0,$$

or $\Delta_1^2 (x_2 - x_3)(x_2 - x_4)(x_3 - x_4) - \Delta_2^2(\ldots) + \Delta_3^2(\ldots) - \Delta_4^2(\ldots) = 0,$

or $(2.3)(2.4)(3.4)\Delta_1^2 - (1.3)(1.4)(3.4)\Delta_2^2$
$+ (1.2)(1.4)(2.4)\Delta_3^2 - (1.2)(1.3)(2.3)\Delta_4^2 = 0.$

6134. (By T. Cotterill, M.A.)—Let aa', bb', cc' be the opposite vertices of four co-planar lines: then, if three systems of conics pass respectively through $(bb'cc')$, $(cc'aa')$, $(aa'bb')$; prove (1) that the invariant $\Theta\Theta' - \Delta\Delta'$ of two conics, one from each system, vanishes, and the double chord of intersection is harmonic to the third system of conics; and (2) give the reciprocal of these properties.

I. *Solution by* J. J. Walker, M.A.

1. Suppose aa', bb' to meet in o, and let cc', $c'o$, oc be x, y, z respectively; then, without any loss of generality, bc', ac' may be taken as $x+y$, $x-y$, and $b'c$, ac as $x+z$, $x-z$. The three systems of conics will then be

$$S_1 \equiv x^2 + yz + kzx + kxy, \quad S_2 \equiv x^2 - yz + k'zx - k'xy,$$
$$S_3 \equiv (1-k')x^2 + k'y^2 - z^2.$$

For the system S_3S_2, $\Delta = (k''-1)k''$, $\Delta' = 2(k'^2-1)$,

$$\Theta = -2k'', \quad \Theta' = (1-k'')(k'^2-1), \text{ so that } \Theta\Theta' - \Delta\Delta' \equiv 0.$$

Again, bb', the common chord of the systems S_3, S_1, is $y+z$, and the pole of this line with respect to S_2 is determined by

$$2x' - k'y' + k'z' = 0, \quad k'x' + z' = y' - k'x',$$

a point on $y - z = 0$, which is aa', the chord common to the systems S_3, S_2.

2. Let $\Delta_1, \Delta_1', \Theta_1, \Theta_1'$ refer to the conics reciprocal to the general conics $ax^2 + \ldots$, $a'x^2 + \ldots$, then it is readily verified that (to a factor *près*)

$$\Delta_1 = \Delta^2, \quad \Delta_1' = \Delta'^2, \quad \Theta_1 = \Delta\Theta', \quad \Theta_1' = \Delta'\Theta;$$

so that $$\Theta_1\Theta_1' - \Delta_1\Delta_1' = \Delta\Delta'(\Theta\Theta' - \Delta\Delta').$$

Hence, if AA', BB', CC' are three pairs of straight lines joining four co-planar points, and if three systems of conics touch respectively (BB'CC'), (CC'AA'), (AA'BB'), the invariant $\Theta\Theta' - \Delta\Delta'$ of two conics, one from either of two of the systems, vanishes; and the polar of the point of intersection of the pair of tangents common to two systems, with respect to any one of the third system passes through the intersection of the tangents common to that third system and either of the former two.

II. *Solution by* Professor WOLSTENHOLME, M.A.

1. Let the four lines be $x \pm y \pm z = 0$; and let (0, 1, 1), (0, 1, −1), (1, 0, 1), (−1, 0, 1), (1, 1, 0), (1, −1, 0) be the coordinates of a, a', b, b', c, c'; then the equation of a conic through $aa'bb'$ will be

$$x^2 + y^2 - z^2 + 2hxy = 0 \equiv S,$$

and that of a conic through $aa'cc'$ will be

$$x^2 - y^2 + z^2 + 2gzx = 0 \equiv S',$$

and the discriminant of $kS + S'$ is

$$(k+1)(k-1)^2 + (k-1)g^2 - (k-1)k^2h^2;$$

so that $\Delta = 1 - h^2$, $\Theta = h^2 - 1$, $\Theta' = g^2 - 1$, $\Delta' = 1 - g^2$,

and the true invariant relation $\Theta\Theta' = \Delta\Delta'$ is always satisfied.

The third system of conics is

$$-x^2 + y^2 + z^2 + 2fyz = 0,$$

and the common chord of the two former is $x = 0$, $y^2 = z^2$,

and the range on $x = 0$, formed by $y^2 = z^2$, $y^2 + z^2 + 2fyz = 0$, is harmonic.

Otherwise:—if a, a' be projected into circular points, bb', cc' will be two pairs of anti-points, or we may take b, b' for the real, and c, c' for the impossible foci of a conic, say $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; then a conic through $aa'bb'$ will be a circle $x^2 + y^2 - 2ky = a^2 - b^2$,

and a conic through $aa'cc'$ will be a circle

$$x^2 + y^2 - 2hx + a^2 - b^2 = 0;$$

and these circles cut at right angles; and, in that case, the harmonic envelope degenerates into a point-pair (the centres of the circles), and the relation $\Theta\Theta' = \Delta\Delta'$ is satisfied. The conics through $bb'cc'$ are the system of rectangular hyperbolas $x^2 - y^2 + 2fxy = a^2 - b^2$, and divide the segment whose extremities are the circular points aa' harmonically.

2. If aa', bb', cc' be the three pairs of straight lines through the four points of a quadrangle, and two systems of conics be drawn touching $aa'bb'$ and $aa'cc'$ respectively, the invariant relation $\Theta\Theta' = \Delta\Delta'$ will be satisfied between any two conics, one from each system, and the two common tangents a, a' will be harmonically conjugate to any conic touching $bb'cc'$. The invariant relation is unaltered, since it is the condition both that the harmonic locus degenerate into a line-pair, and that the harmonic envelope degenerate into a point-pair.

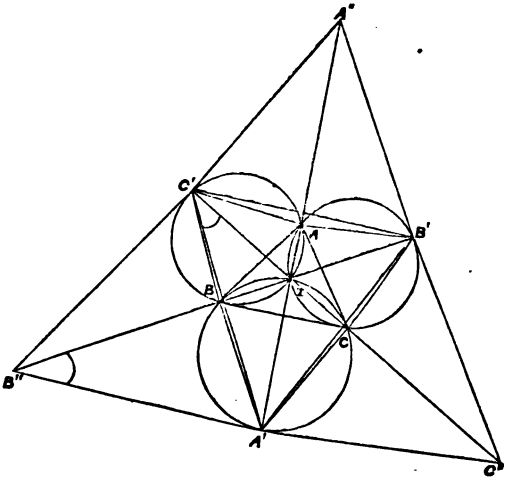
5782. (By T. MITCHESON, B.A., L.C.P.)—If equilateral triangles be drawn on the sides of a triangle, prove that the lines that join the vertices of these triangles with the opposite corners of the original triangle, and

the circles drawn round these triangles, meet at a point where the sides of the triangle subtend equal angles.

5932. (By Professor Cochez.) — Connaissons les sommets de trois triangles équilatéraux construits sur les côtés d'un triangle, construire ce triangle.

Solution by Professor Cochez; H. N. Capel, LL.B.; *and others.*

(Quest. 5782.) Soit ABC un triangle quelconque; BCA', CAB', ABC' les triangles équilatéraux construits sur les côtés de ce triangle.



Les triangles ACC', BAB' sont égaux comme ayant un angle C'AC = B'AB compris entre côtés égaux, donc CC' = BB'; on démontrerait, de même que BB' = AA'. Donc AA' = BB' = CC'.

Circonscrivons des cercles aux triangles BCA', BAC'. Soit I leur second point d'intersection. Si l'on joint AI, BI, CI, les angles BIC et AIB sont égaux à 120°; il en sera de même de l'angle AIC. Le cercle circonscrit à AB'C passera également en I.

Menons IA', IB', IC'. Les angles consécutifs A'IC, CIB', AIB' valent 60°. Donc IA', IB', IC' sont les prolongements de AI, BI, CI. Donc les trois droites et les trois cercles se coupent en un même point.

De ce point I on voit également sous un angle de 120° les côtés du triangle A'B'C'. Si on effectuait la même construction sur les côtés du triangle A'B'C', les droites A'A'', B'B'', C'C'' sont aussi égales et se coupent en un certain point duquel on voit les côtés du triangle A'B'C' et ceux de ABC sous un angle de 120°. Ce point coincide donc avec I. Donc les droites AA'A'', BB'B'', CC'C'' sont concourrantes.

(Quest. 5932.) Ce théorème permet de résoudre le problème dans la Quest. 5932 [which is, in fact, identical with Quest. 5811, whereof a solution is given on p. 60 of this volume of the *Reprint*].

Soient A', B', C' les points donnés, joignons ces points et sur les droites

A'B, C'B', A'C' construisons des triangles équilatéraux. Joignons A''A', B''B', C''C'. Ces droites se coupent en I. Si on prend les milieux A, B, C de ces droites, et si on les joint, on a le triangle cherché ABC.

En effet, les triangles A'BB'', A'CC' sont égaux, car CC' = BB'' comme moitiés de droites égales. A'C' = A'B'' par construction, et le quadrilatère A'IC'B'' étant inscriptible puisque l'angle en B'' vaut 60° et l'angle C IA' 120°, donc les angles A'C'C et A'B''B sont égaux, donc AB' = A'C. Mais B''A'B = 60°, il en est dès lors de même de BA'C. Donc BA'C est le triangle équilatéral construit sur BC. On démontrerait de même que AB'C et BAC' sont équilatéraux. Donc ABC est bien le triangle cherché.

6060. (By J. Young, B.A.)—Prove that any one of the three following equations is derivable from the other two :—

$$A \equiv a^2(x^2+xy+y^2) - axy(x+y) + x^2y^2 = 0,$$
$$B \equiv a^2(y^2+yz+z^2) - ayz(y+z) + y^2z^2 = 0,$$
$$C \equiv a^2(z^2+zx+x^2) - azx(z+x) + z^2x^2 = 0.$$

Solution by J. O'Regan ; I. H. Turrill, M.A. ; *and others.*

We have $\dfrac{z^2A - x^2B}{ay(z-x)} = a(xy+yz+zx) - xyz = \dfrac{x^2B - y^2C}{az(x-y)}$;

therefore, if any two of A, B, C vanish, so likewise must the third.

5883. (By Prof. Tanner, M.A.)—Given k, a positive proper fraction, and the series $k'_1, k'_2, \ldots k_1, k_2 \ldots$ formed from it by successive application of the equations $(1+k'_n)(1+k_{n-1}) = 2$, $k_n^2 + k_n'^2 = 1$; show that, when n is indefinitely increased, $(1+k'_n)^{2^{n-1}}$ tends to the limit 1.

Solution by the Proposer.

To prove this it is only necessary to show that, when n is indefinitely increased, $2^{n-1}k'_n$ tends to vanish. [Now, from the given equations, we have

$$k'^2_{n-1} = \frac{4k'_n}{(1+k'_n)^2} \quad \text{or} \quad k'_n = \frac{(1+k'_n)^2}{4} \cdot k'^2_{n-1}.$$

Thus k'_n is positive. Also, since k'_n is less than 1, the coefficient of k'^2_{n-1} is less than 1 ; therefore $k'_n < k'^2_{n-1}$.

Hence we have in succession

$$0 < k'_n < k'^2_{n\ 1} < k'^4_{n\ 2} < \ldots < k'^{2^n} ; \text{ therefore } 0 < 2^n k'_n < 2^n k'^{2^n}.$$

Now the right-hand expression lends to the limit 0 as n is indefinitely increased (for, k' being less than 1, mk'^m becomes 0 when $m = \infty$). Hence the same must be true of $2^n k'_n$, and the theorem is proved.

[In DURÈGE'S *Elliptic Functions*, p. 212, 2nd ed., the author seems to assume that a^m tends to the value 1 if a does so. But, from the above, this is evidently insufficient; we require also that $(a-1)\,m$ should vanish, and this is not always the case when m is infinite. For example, we may cite the well-known instance, $(1+kx)^{x^{-1}}$, which, when x becomes infinitesimal, tends to the limit e^k, although $1+kx$ tends to the limit 1.]

6167. (By Dr. HOPKINSON, F.R.S.)—Two magneto-electric machines, with permanent inducing magnets, are used for the transmission of power, the first machine driven by an engine with uniform velocity, and producing a current which is passed through the second machine, the second machine acting as a magneto-electric engine. When the second machine is standing, but the current of the first is passed through its coil, the first machine is found to absorb a power W from the engine. Neglecting the loss due to friction, to disruptive discharges, and to short circuiting of the coils; prove (1) that the maximum power obtainable from the second machine is $\frac{1}{4}$W, and (2) that then the first machine absorbs $\frac{1}{2}$W from the engine.

Solution by Professor G. CAREY FOSTER, F.R.S.

Let the work taken in by the first machine be, when the second is standing, $= W_1$, and when the second is running, $= W'_1$; the work given out by the second machine $= W_2$; the electromotive force of first machine, when the second is standing, $= E_1$, and when the second is running, $= E'_1$; the inverse electromotive force of second machine $= E_2$; and the resistance of the circuit $= R$. Among these quantities we have, by JOULE'S law of the energy of electric currents, the equations

$$W_1 = \frac{E_1^2}{R}, \qquad W'_1 = \frac{E'_1\,(E'_1 - E_2)}{R}, \qquad W_2 = \frac{(E'_1 - E_2)\,E_2}{R}.$$

The two factors in the numerator of the last expression have the constant sum E'_1; therefore W_2 has a maximum value when they are both equal to $\frac{1}{2}E'_1$. Hence the maximum of W_2 is always equal to $\frac{1}{4}W'_1$, as stated in the problem; but to determine the relation between W'_1 and W_1, a further condition is required. We may consider three special cases:—(1) When the speed of the first machine is the same, whether the second is standing or running. (This condition might be practically realised by help of a governor, and might be nearly realised without one, if the work taken in by the magneto-electric machine were a small fraction of that given out by the prime mover.) (2) The work taken in by the first machine is constant, or $W_1 = W'_1$. (This would be approximately the case when there is no governor, and the first machine takes in nearly all the useful work given out by the prime mover.) (3) The strength of the current is constant, or $E_1 = E'_1 - E_2$.

Case 1.—Here $E_1 = E'_1$, and therefore, when W_2 is a maximum, $E_2 = \frac{1}{2}E_1$, and therefore $W'_1 = \frac{1}{2}W_1$. This is evidently the case contemplated in the problem.

Case 2.—Since $W_1 = W'_1$, the maximum of W_2 becomes $= \frac{1}{2}W_1$. In this case we have $E_2 = \frac{E_1}{\sqrt{2}}$, since $E_2 = \frac{1}{2}E'_1$, and $E_1^2 = E'_1(E'_1 - E_2)$.

Case 3.—Here, since $E'_1 - E_2 = E_2$ as the condition for the maximum of W_2, and also by supposition $= E_1$, we have $W_2 = W_1$, and $W'_1 = 2W_1$.

5494. (By Professor TAIT.)—Determine the number of *distinct* knots with 8 crossings, and figure one form of each.

Remarks by the PROPOSER.

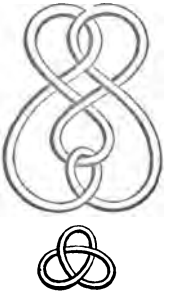
In a paper which appeared in the *Transactions* of the Royal Society of Edinburgh (1877), I have shown *how* this can be done. I had leisure to solve it completely for 3, 4, 5, 6, 7 crossings, but for the 8-fold knot (at least by *my* process) over 4000 separate cases have to be investigated, and I cannot spare the requisite time. I have shown that the number of knots (distinct from one another) with these numbers of crossings, is as follows:—

Number of crossings... 3, 4, 5, 6, 7, 8, ...

Number of species ... 1, 1, 2, 4, 8, ..., ...

It is not likely that the geometric series will hold for another term, but I am desirous of knowing what the number for 8 is. When 9 is in question, the selection has to be made from among over 40,000!

Here is one 8-fold knot as a specimen. It is peculiar, inasmuch as it is what I call *amphicheiral*, that is to say, it can (if made of cord) be changed into its own perversion or image in a plane mirror. This property the annexed *trefoil* does not possess.



6156. (By Professor TOWNSEND, F.R.S.)—The axis of principal moment of any system of forces in a common space being supposed taken with respect to a variable point on any fixed right line in the space; show that it generates a skew paraboloid, whose axis of figure is the common perpendicular to the directive right line, and to the central axis of the system.

Solution by J. J. WALKER, M.A.

Let OX ($=a$), the common perpendicular to the directive line OL and the central axis, OZ parallel to the latter, and OY perpendicular to OX, OZ, be taken as axes of coordinates; and let the equation to OL be $qy - z = 0$. Then, if R, pR are the magnitudes of the resultant force and of the minimum principal moment respectively; the axis of principal moment relative to a point $(0, y', z')$ on OL will be the line joining that point with $x'' = \mathrm{R}y'$, $y'' = y' + a\mathrm{R}$, $z'' = z' + p\mathrm{R}$, and its equation will therefore be $x : \mathrm{R}y' = y - y' : a\mathrm{R} = z - qy' : p\mathrm{R}$. Eliminating y', we have $(aq-p)^2 x = (qy-z)(az-py)$, a hyperbolic paraboloid, having OX as its axis, for the surface containing the axis of principal moment at points on OL. [Mr. WALKER remarks that this theorem formed the second part of Prof. MINCHIN'S Quest. 5566, of which a Quaternion proof by him is given on p. 40 of Vol. XXXII. of our *Reprints*.]

5762. (By Professor SYLVESTER, F.R.S.)—If $f(x, y)$ is a quantic of n dimensions in x, y, prove that $(n-1)\left(\frac{df}{dx}\right)^2 - nf\frac{d^2f}{dx^2}$ is a negative numerical multiplier of y^2 into the Hessian of f.

6077. (By W. J. C. SHARP, M.A.)—If any equation have three equal roots, prove that (1) the Hessian will have the same root as a double root; or generally, (2) that if any equation have a p-ple root, the same quantity is a $2(p-1)$-ple root of the Hessian.

Solution of both Questions by W. J. CURRAN SHARP, M.A.

By the known properties of homogeneous functions, we have

$$n(n-1)f = x^2\frac{d^2f}{dx^2} + 2xy\frac{d^2f}{dx\,dy} + y^2\frac{d^2f}{dy^2}, \text{ and } (n-1)\frac{df}{dx} = x\frac{d^2f}{dx^2} + y\frac{d^2f}{dx\,dy};$$

therefore
$$(n-1)^2\left(\frac{df}{dx}\right)^2 - n(n-1)f\frac{d^2f}{dx^2}$$
$$= \left(x\frac{d^2f}{dx^2} + y\frac{d^2f}{dx\,dy}\right)^2 - \frac{d^2f}{dx^2}\left(x^2\frac{d^2f}{dx^2} + 2xy\frac{d^2f}{dx\,dy} + y^2\frac{d^2f}{dy^2}\right)$$
$$= -y^2\left\{\frac{d^2f}{dx^2}\frac{d^2f}{dy^2} - \left(\frac{d^2f}{dx\,dy}\right)^2\right\},$$

therefore
$$(n-1)\left(\frac{df}{dx}\right)^2 - nf\frac{d^2f}{dx^2} = -\frac{y^2}{n-1}\mathrm{H}.$$

From this it follows that, if the equation $f = 0$ have p roots each $= a$, the Hessian H will have $2(p-1)$ roots each $= a$.

If, then, the equation have $(n-1)$ roots each $= a$, the Hessian will be $(ac-b^2)(x-a)^{2(n-2)}$; by comparing which with ordinary form, the requisite conditions may be obtained. Thus, the conditions that quartic $ax^4 + 4bx^3 + \&c. = 0$ may have three equal roots, are

$$\frac{ad-bc}{2(ac-b^2)} = \frac{ae+2bd-3c^2}{3(ad-bc)} = \frac{3(be-cd)}{ae+2bd-3c^2} = \frac{2(ce-d^2)}{be-cd},$$

which give $T=0$ and $S=0$, the necessary and sufficient conditions, as the above equations are not all independent.

[Prof. SYLVESTER remarks that the effect of the theorem in Quest. 5762 is to show that, in his extension of NEWTON'S Rule, the double series (of which the double permanences are to be counted in order to determine a superior limit to the number of real roots included between any two given quantities) may be written as follows:

$$fx,\ f'x,\ f''x,\ \ldots\ldots,\ f^{(n-1)}x,\ f^{(n)}x;$$
$$-1,\ Hfx,\ Hf'x,\ \ldots\ldots,\ Hf^{(n-2)}x,\ -1;$$

as, for example, for a biquadratic the series may be taken as

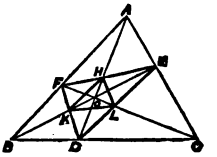
$$fx,\ f'x,\ f''x,\ f'''x,\ f''''x;$$
$$-1,\ Hfx,\ Hf'x,\ Hf''x,\ -1;$$

where $H\phi(x)$, in general, means the Hessian of ϕx, regarded as a homogeneous function of x and 1. Since the $(n-2)^{\text{th}}$ derivative of f is a quadratic in x, it follows that the last of the H's is always a constant. This the PROPOSER believes to be a most important transformation of the theorem as it has been hitherto given.]

4208. (By Professor MONCK, M.A.)—Three lines AD, BE, CF are drawn from the vertices of a triangle ABC through a point O to the opposite sides; prove that $2\triangle ABC : \triangle DEF = AO \cdot BO \cdot CO : DO \cdot EO \cdot FO$.

Solution by R. TUCKER, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

$$\frac{AO}{DO} = \frac{\triangle AOC}{\triangle DOC} = \frac{\triangle AOC}{\triangle BOC} \cdot \frac{\triangle BOC}{\triangle DOC} = \frac{AF}{BF} \cdot \frac{BC}{DC};$$



also $$\frac{BO}{EO} = \frac{AC \cdot BD}{AE \cdot CD},$$

and $$\frac{CO}{FO} = \frac{AB \cdot CE}{BF \cdot AE};$$

therefore $$\frac{AO \cdot BO \cdot CO}{DO \cdot EO \cdot FO} = \frac{AB \cdot BC \cdot CA \cdot AF \cdot BD \cdot CE}{(AE \cdot CD \cdot BF)^2} = \frac{AB \cdot BC \cdot CA}{AE \cdot CD \cdot BF}$$

$= $ (by Quest. 4184, *Reprint*, Vol. XX., p. 104) $\dfrac{abc}{2R\triangle DEF} = \dfrac{2\triangle ABC}{\triangle DEF}$.

[Let AD, BE, CF meet EF, FD, DE in H, K, L; and let Δ be the area of the primitive triangle ABC, and Δ_2 the area of the second derivative triangle HKL; then, putting

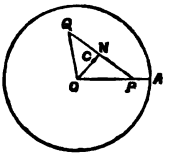
$$AO \cdot BO \cdot CO = p_1, \text{ and } DO \cdot CO \cdot EO = p_2,$$

it may be readily shown that $2^2\Delta : \Delta_2 = p_1 : p_2$; and proceeding, *seriatim*, to the nth derivative triangle, we shall have $2^n\Delta : \Delta_n = p_1 : p_2$. The foregoing solution furnishes also a solution of Quests. 5899, 6003.]

6132. (By Elizabeth Blackwood.)—If O be the centre of a given circle of radius r, P and Q random points in it, and C the centroid of the triangle OPQ; show that the average value of OC is $\frac{128r}{135\pi}$.

Solution by Prof. Seitz; A. Martin, M.A.; *and others.*

Let $OA=r$, $OP=x$, $OQ=y$, $OC=z$, $\angle POQ=\theta$. Then we have $z=\frac{1}{3}(x^2+y^2+2xy\cos\theta)^{\frac{1}{2}}$; the limits of θ are 0 and π; of x, 0 and r; and of y, 0 and x. Hence the average value of OC is



$$\int_0^\pi\int_0^r\int_0^x z\,d\theta\,x\,dx\,y\,dy \div \int_0^\pi\int_0^r\int_0^x d\theta\,x\,dx\,y\,dy$$

$$=\frac{8}{3\pi r^4}\int_0^\pi\int_0^r\int_0^x (x^2+y^2+2xy\cos\theta)^{\frac{1}{2}}\,d\theta\,x\,dx\,y\,dy$$

$$=\frac{2}{9\pi r^4}\int_0^\pi\int_0^r \{8\cos\tfrac{1}{2}\theta+40\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-48\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-1 + 3\cos 2\theta-6\sin^2\theta\cos\theta\log(1+\sec\tfrac{1}{2}\theta)\}\,d\theta\,x^4\,dx$$

$$=\frac{2r}{45\pi}\int_0^\pi \{8\cos\tfrac{1}{2}\theta+40\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-48\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-1 + 3\cos 2\theta-6\sin^2\theta\cos\theta\log(1+\sec\tfrac{1}{2}\theta)\}\,d\theta=\frac{128r}{135\pi}.$$

6191. (By Professor Crofton, F.R.S.)—Show that the common centre of gravity of any two arcs AB, A′B′ of the same circle, lies in the diameter perpendicular to the line joining the middles of the chords AA′, BB′.

Solution by C. J. Monro, M.A.; Morgan Jenkins, M.A.; *and others.*

In other words, if a quadrilateral is inscribed in a circle, the bisector of the diagonals is perpendicular to the mass-vector of arcs cut off by opposite sides, the origin being taken at the centre. Let r be the radius; m the mass of unit arc; α, β, α', β' vectors of A, B, A′, B′ from the centre. Since $\int e^{i\theta}\,d\theta=-ie^{i\theta}$, the mass-vector of opposite arcs is

$$\pm mri(\alpha-\beta+\alpha'-\beta'),$$

perpendicular to $-\frac{1}{2}(\alpha+\alpha')+\frac{1}{2}(\beta+\beta')$ bisecting the diagonals.

6047. (By Donald McAlister, B.A., B.Sc.)—Prove the following rule for finding the centre of inertia of the surface of a right cone cut off

by an oblique plane:—Make a plane section parallel to the base and at two-thirds of its distance from the vertex; let the axis cut this plane in A, and let B be its centre; then C, the point required, lies in AB produced, and AB : AC = 2 : 3.

Solution by R. Leidhold, M.A.; T. R. Terry, M.A.; *and others.*

Let a be the vertex and b the centre of the base. Consider the surface as made up of an infinite number of small triangles, having a for vertex, and for bases the sides of the polygon, which ultimately becomes the ellipse whose centre is b. Thus, the centre of inertia must be in the plane two-thirds of the way from a to the base.

Next, take a plane perpendicular to the axis. All the surface is equally inclined to this plane; therefore the centre of inertia of the surface projects orthogonally into the centre of inertia of the area of the projection; that is, into the projection of b; but $aB = \frac{2}{3}ab$; therefore $AB = \frac{2}{3}AC$.

3916. (By S. Watson.)—If O, O_1, O_2, O_3 be the centres of the inscribed and escribed circles of a triangle ABC; show that the sum of the squares of the distances of the centroids of the triangles BOC, COA, AOB, BO_1C, CO_2A, AO_3B from the centroid of the triangle ABC, is

$$\tfrac{2}{9}(ab+bc+ca)+\tfrac{1}{3}D(D-r).$$

Solution by Professor Wolstenholme, M.A.

The distance between the centres of inertia of the triangles ABC, BOC, $=\frac{1}{3}OA$; and similarly for each of the others; hence, sum of squares

$$= \tfrac{1}{9}(AO^2+AO_1^2+\ldots) = \tfrac{1}{9}[(4R\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C)^2+(4R\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C)^2+\ldots]$$

$$= \tfrac{1}{9}[8R^2(1+\cos B\cos C)+\ldots+\ldots]$$

$$= \tfrac{8}{9}R^2(\sin B\sin C+\ldots)+\tfrac{8}{9}R^2+\tfrac{8}{9}R^2[\cos(B-C)+\ldots]$$

$$= \tfrac{2}{9}(bc+ca+ab)+\tfrac{8}{9}R^2(3-\cos A-\cos B-\cos C)$$

$$= \tfrac{2}{9}(bc+ca+ab)+\tfrac{8}{9}R^2\left(2-\frac{r}{R}\right) = \tfrac{2}{9}(bc+ca-ab)+\tfrac{4}{9}(D^2-Dr).$$

5745. (By S. Tebay, B.A.)—A heavy egg-shaped shell, whose interior surface is perfectly smooth, is filled with incompressible fluid and placed upon a smooth horizontal plane. If it be twirled about a vertical axis through their common centre of gravity, determine the motion.

Solution by the PROPOSER.

Using the customary notation, Mr. WOOLHOUSE (*Gentleman's Diary* for 1838) has found, for the motion of any solid of revolution,

$$\frac{d\phi}{dt} - \cos\theta \frac{d\omega}{dt} = c, \quad c\cos\theta - \frac{A}{C}\sin^2\theta \frac{d\omega}{dt} = c',$$

$$A\left\{\sin^2\theta\left(\frac{d\omega}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2\right\} + M\left\{\left(\frac{dz}{dt}\right)^2 + 2gz\right\} = c'';$$

c, c', c'' being constants depending on initial motion, M the mass of the solid, z the perpendicular from the centre of gravity on the plane, and ω the inclination to x of the line from the foot of z to the point of contact. For M write $M+m$, m being the mass of the fluid; then

$$A\left\{\sin^2\theta\left(\frac{d\omega}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2\right\} + (M+m)\left\{\left(\frac{dz}{dt}\right)^2 + 2gz\right\} = c''.$$

If m be small in comparison with M, the motion corresponds to that of a hard-boiled egg. If m be large compared with M, we have

$$\left(\frac{dz}{dt}\right)^2 + 2gz = c'';$$

and as no motion is communicated to m, $\frac{dz}{dt} = 0$, and z is constant.

[This question took its rise from Col. CLARKE's ingenious problem (5700), to find, *by a mathematical process*, whether a boiled egg is hard or soft, two solutions of which have been given in *Reprint*, Vol. XXX., pp. 30, 42. Mr. TEBAY's investigation, as given above, has been objected to "as no solution at all," inasmuch as, besides other defects, "there is no quality of fluids taken into consideration."]

5942. (By C. LEUDESDORF, M.A.)—Show that the equation of the segment of a circle which passes through (x_1, y_1), (x_2, y_2), and contains an angle θ, is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = \pm\cot\theta \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$.

Solution by G. TURRIFF, M.A.; C. K. PILLAI, M.A.; *and others.*

Let (α, β) be a point on the segment; then the equations of the lines joining (α, β) with (x_1, y_1), (x_2, y_2) are

$$y-\beta = \frac{y_1-\beta}{x_1-\alpha}(x-\alpha), \text{ and } y-\beta = \frac{y_2-\beta}{x_2-\alpha}(x-\alpha);$$

therefore $$\pm\tan\theta = \left\{\frac{y_1-\beta}{x_1-\alpha} - \frac{y_2-\beta}{x_2-\alpha}\right\} \div \left\{1 + \frac{(y_1-\beta)(y_2-\beta)}{(x_1-\alpha)(x_2-\alpha)}\right\};$$

hence we have $$(x_1-\alpha)(x_2-\alpha)+(y_1-\beta)(y_2-\beta)$$
$$= \pm\cot\theta\{(y_1-\beta)(x_2-\alpha)-(y_2-\beta)(x_1-\alpha)\} = \&c.$$

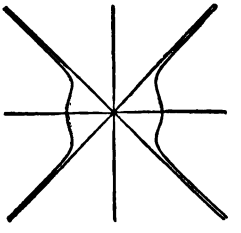
6142. (By W. H. H. Hudson, M.A.)—Trace the curve

$$r(\cos 2\theta)^{\frac{1}{2}} = 2a\cos^2\theta,$$

and find its points of inflexion.

Solution by G. Heppel, M.A.; C. Bickerdike, B.A.; *and others.*

It is clear that the curve is included between the lines $\theta = \frac{1}{4}\pi$ and $\theta = -\frac{1}{4}\pi$, that these lines are asymptotes, and that there is a similar branch on the negative side of the origin. To find a point of inflexion, put $v = \cos 2\theta$; then we have



$$\frac{dv}{d\theta} = -2(1-v^2)^{\frac{1}{2}}, \quad \frac{d^2v}{d\theta^2} = -4v,$$

$$\frac{a}{r} = au = \frac{v^{\frac{1}{2}}}{1+v}, \quad a\frac{du}{dv} = \frac{1-v}{2(1+v)^2v^{\frac{1}{2}}},$$

$$a\frac{d^2u}{dv^2} = \frac{3v^2-6v-1}{4(1+v)^3v^{\frac{3}{2}}},$$

$$\frac{d^2u}{d\theta^2} = \frac{d^2u}{dv^2}\cdot\left(\frac{dv}{d\theta}\right)^2 + \frac{du}{dv}\cdot\frac{d^2v}{d\theta^2};$$

and at a point of inflexion $u + \frac{d^2u}{d\theta^2} = 0$; therefore $8v^2 - 5v - 1 = 0$,

whence $v = \cos 2\theta = \frac{1}{16}(5+\sqrt{57})$, and $\theta = 19°\,10'$.

From 0° to 20° the curve is almost coincident with (being just outside) a circle whose radius is $2a$.

[Since $r(\cos 2\theta)^{\frac{1}{2}} = 2a\cos\theta = a(1+\cos 2\theta)$, or $r = a(\sec 2\theta)^{\frac{1}{2}} + a(\cos 2\theta)^{\frac{1}{2}}$, each radius of the curve is an arithmetic mean between the corresponding radii of the rectangular hyperbola $r = 2a(\sec 2\theta)^{\frac{1}{2}}$, and the lemniscate $r = 2a(\cos 2\theta)^{\frac{1}{2}}$; hence, from the figure, it is clear that there must be points of inflexion, and they may be accordingly found in the usual way.]

6136. (By the Rev. W. A. Whitworth, M.A.)—Each of two urns contains n cards numbered successively 1, 2, 3, ... n. I draw a card from each urn and receive a number of shillings equal to the difference between the numbers on the two cards drawn. I repeat this operation, again receiving the difference in shillings between the two numbers which I draw; and so on, until at the n^{th} operation the cards are exhausted. Show that my expectation is worth $\frac{1}{3}(n^2-1)$ shillings.

Solution by the Rev. J. L. Kitchin, M.A.; G. Heppel, M.A.; *and others.*

The number of cases where the card r in the first urn comes out in combination with the card s in the second, is $(n-1)!$, and the corresponding money value is $(n-1)!\,(r \sim s)$. Hence the money value of the total of cases, where r stands first, is

$$\begin{aligned}(n-1)!\,\{0+1+2+3+\ldots+(r-1)!+1+2+3+\ldots+(n-r)!\}\\ = (n-1)!\,\{\tfrac{1}{2}r(r-1)!+\tfrac{1}{2}(n-r+1)(n-r)\}\\ = \tfrac{1}{2}(n-1)!\,\{n^2+n-2(n+1)r+2r^2\}.\end{aligned}$$

Hence, giving r all values from 1 to n, the total of all money values is

$$\tfrac{1}{2}(n-1)!\left\{n^3+n^2-n(n+1)^2+\tfrac{1}{3}n(n+1)(2n+1)\right\} = \tfrac{1}{2}(n+1)!(n-1);$$

and of these values I receive some one out of $n!$ sets; therefore my expectation is worth $\frac{1}{2}(n^2-1)$ shillings.

6043. (By the EDITOR.)—Find the locus of the centres of circles tangential to a parabola and the tangent at its vertex.

I. *Solution by* Professor WOLSTENHOLME, M.A.

The equation of the locus is

$$y^4-y^2(2x^2+18ax+27a^2)+x^3(x+2a)=0,$$

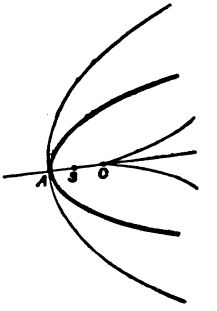
the equation of the parabola being

$$y^2=4a(x+2a).$$

There is a cusp at the origin, which is the centre of curvature of the vertex of the parabola, and there are the two parabolic asymptotes

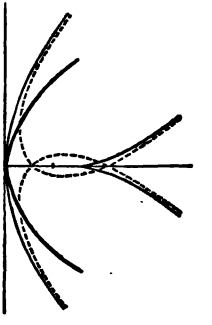
$$(\pm y+x+\tfrac{3}{2}a)^2=2a(\mp y+x+\tfrac{3}{2}a).$$

The form of the curve is indicated by the fainter lines; where A, S are the vertex and focus of the given parabola, and C is the centre of curvature at the vertex.



II. *Solution by* G. HEPPEL, M.A.; CHRISTINE LADD; *and others.*

In the figure the thick lines represent the parabola and its tangent, the thin lines are the required locus, and the dotted lines are parabolic asymptotes.



It is clear from the conditions of the question that the axis of x is a part of the locus; and, to find the rest, let $y^2=4ax$ be the parabola; (x, y) the centre of the circle; and (h, k) the point where it touches the parabola; then, to eliminate h, k, $\frac{dk}{dh}$, we have the following equations:—

$$(x-h)^2+(y-k)^2=x^2, \qquad k^2=4ah,$$

$$x-h+(y-k)\frac{dk}{dh}=0, \qquad k\frac{dk}{dh}=2a.$$

The result is $2y^2=2x^2+10ax-a^2\pm a^{\frac{1}{2}}(a+4x)^{\frac{3}{2}}$,

or $\quad y^4-y^2(2x^2+10ax-a^2)-x(2a-x)^3=0.$

From the second form we see that, when x is less than $2a$, there is one

positive value of y^2; and when x is greater than $2a$, there are two. Also, if $x = 2a$, $y = 0$, and $\frac{dy}{dx} = \frac{0}{0}$; hence there is a cusp at this point.

From the first form, expanding y^2 in descending powers of x, we have

$$y^2 = x^2 \pm 4a^{\frac{1}{2}}x^{\frac{3}{2}} + 6ax \pm \tfrac{3}{2}a^{\frac{3}{2}}x^{\frac{1}{2}} - \tfrac{1}{2}a^2 \pm \tfrac{3}{8}a^{\frac{1}{2}}x^{-\frac{1}{2}} + \&c.;$$

therefore $$\pm y = x \pm 2a^{\frac{1}{2}}x^{\frac{1}{2}} + \tfrac{1}{2}a \mp \tfrac{1}{4}a^{\frac{3}{2}}x^{-\frac{1}{2}} + \tfrac{1}{8}a^2x^{-1} + \&c.,$$

therefore $$\pm y - x - \tfrac{1}{2}a = \pm 2a^{\frac{1}{2}}x^{\frac{1}{2}} \mp \tfrac{1}{4}a^{\frac{3}{2}}x^{-\frac{1}{2}} + \tfrac{1}{8}a^2x^{-1} + \&c.,$$

therefore $(\pm y - x - \frac{1}{2}a)^2 = 4ax - a^2$ are parabolic asymptotes.

These have their vertices at a point $x = \frac{1}{2}a$, $y = 0$, and have their axes inclined at an angle of 45° to the axis of x. The latus rectum of each is $a\sqrt{2}$. They cut the axis of x again at a point $x = \frac{5}{2}a$.

6184. (By G. Heppel, M.A.) — Find the number of terms in the expansion of the k^{th} power of the sum of n different quantities.

Solution by the Rev. J. L. Kitchin, M.A.; Christine Ladd; *and others.*

The number of terms will be the number of combinations of the n things $a, b, c, \ldots$ (including repetitions) taken k together; and this is $N_{n,k} = \dfrac{(n+k-1)!}{(n-1)!\,k!}$, a well-known result.

6048 & 6086. (By J. Hammond, M.A.)—Prove that the n^{th} convergent to $2 + \dfrac{1}{2+}\dfrac{1}{2+}\dfrac{1}{2+}\ldots$ is $\dfrac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^n - (1-\sqrt{2})^n}$;

and to $2 - \dfrac{1}{2-}\dfrac{1}{2-}\dfrac{1}{2-}\cdots$ is $\dfrac{n+1}{n}$.

Solution by Prof. Tanner, M.A.; R. Knowles, B.A., L.C.P.; &c.

(Quest. 6048.) If u_n represent the n^{th} convergent, we have $u_n = 2 + \dfrac{1}{u_{n-1}}$.

Put $\dfrac{v_{n+1}}{v_n} = u_n$; then the equation becomes $v_{n+1} = 2v_n + v_{n-1}$; giving

$$v_n = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n,$$

and $$u_n = \frac{c_1(1+\sqrt{2})^{n-1} + c_2(1-\sqrt{2})^{n+1}}{c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n}.$$

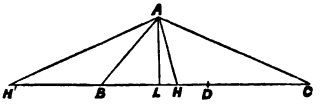
But $u_1 = 2$, so that $c_1 + c_2 = 0$; and this gives the required form.

(Quest. 6086.) The converging fractions in this case are $\frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{5}{2}$, &c.; and the law of formation gives us $a_n = 1 + b_n$; and since $b_1 = 1$, $b_2 = 2$,... $b = n_n$, therefore $a_n = 1 + n$, and $\frac{a_n}{b_n} = \frac{n+1}{n}$.

6178. (By E. Anthony, M.A.)—Let D be the middle of the base BC of a triangle ABC; AL the perpendicular from A on BC; and AH, AH' the bisectors of the angle A; then prove that $DL \cdot HH' = AB \cdot AC$.

Solution by A. L. Selby, M.A.; E. Rutter; *and others.*

$AB \cdot AC = BH \cdot HC + AH^2$
$= BH \cdot HC + HL \cdot HH'$;
and since $P\{BHCH'\}$ is harmonic,
we have $DH \cdot DH' = DC^2$;
therefore $DH \cdot HH' = BH \cdot HC$,
and $AB \cdot AC = DH \cdot HH' + HL \cdot HH'$
$= DL \cdot HH'$.



6116. (By T. Morley, L.C.P.)—Solve the equations
$$x^3 - a = y^3 - b = z^3 - c = xyz.$$

Solution by E. W. Symons, B.A.; T. R. Terry, M.A.; *and others.*

Eliminating y and z, we have
$$(a+b+c)\,x^6 - (2a^2 + ab + ac - bc)\,x^3 + a^3 = 0;$$
$$\therefore \quad 2(a+b+c)\,x^3 = 2a^2 + ab + ac - bc \mp [a^2b^2 + b^2c^2 + c^2a^2 - 2abc\,(a+b+c)]^{\frac{1}{2}}.$$
The values of y and z follow from symmetry.

5272. (By Professor Townsend, F.R.S.)—A solid of revolution of uniform density, turning without friction round a fixed point on its axis of figure, and rolling without sliding on a fixed plane inclined at any angle to the horizon, being supposed to have the velocity of rotation just sufficient to carry it with exhausted energy of motion to its position of

unstable equilibrium against the action of gravity; determine, in finite terms, its time of passage from the opposite position of stable equilibrium to any other on its way.

Solution by the PROPOSER.

Denoting by O the fixed point, by OP and OV the perpendicular and the vertical from it to the plane, by C the centre of inertia of the body, by X its point of contact with the plane in any position, by A and B the positions of X on the line PV which correspond to those of stable and unstable equilibrium of the body, by θ and ϕ the two variable angles APX and COV, by ω the variable velocity of rotation round the instantaneous axis OX, by k the constant arm of inertia round OX, by α the fixed angle POV of inclination of the plane, by β and γ the two constant angles COP and COX, by l the constant distance OC, and by m the mass of the body; then, since, for any two positions of the body, by the equation of *vis viva*, $k^2(\omega^2 - \omega_1^2) = 2gl(\cos\phi - \cos\phi_1)$, since, by the geometry of rotation, $\omega \sin\gamma = \frac{d\theta}{dt}\sin\beta$, and since, by spherical trigonometry, $\cos\phi = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\theta$; therefore, for any two positions of the body, $\frac{d\theta^2}{dt^2} - \frac{d\theta_1^2}{dt^2} = 2\frac{g}{l'}.(\cos\theta - \cos\theta_1)$,

where $$l' = \frac{k^2}{l}.\frac{\sin\beta}{\sin\alpha}.\operatorname{cosec}^2\gamma.$$

Hence, when, as by hypothesis, $\frac{d\theta_1}{dt} = 0$ when $\theta_1 = \pi$, then

$$\frac{d\theta^2}{dt^2} = 2\frac{g}{l'}(1+\cos\theta) = 4\frac{g}{l'}\cos^2\tfrac{1}{2}\theta;$$

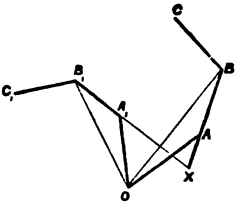
and therefore, by integration in finite terms, supposing θ and t to commence together, $$t = \left(\frac{l'}{g}\right)^{\frac{1}{2}}\log(\tan\tfrac{1}{2}\theta + \sec\tfrac{1}{2}\theta);$$

which accordingly is the required value, in finite terms, for t in terms of θ.

Putting $\theta = \pi$ in the above formula, we get $t = \infty$; which accords with the general property that, under circumstances similar to the above, a position of unstable equilibrium can be reached with exhausted energy of motion by a body or system of bodies against the action of the forces, only at the expiration of an infinite time.

5264. (By the REV. H. G. DAY, M.A.)—Two similar polygons ABCDE..., $A_1B_1C_1D_1E_1$... are similarly situated on lines AB, A_1B_1 (not parallel); show that (1) if they are situated towards the same parts, a point O can be found at which AA_1, BB_1, CC_1, &c. all subtend the same angle; and (2) if towards opposite parts, a point O can be found at which the bisectors of the angles AOA_1, BOB_1, COC_1, coincide.

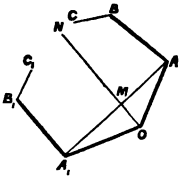
Solution by PROPOSER.



1. Produce BA, B_1A_1 to meet in X; and draw circles round AA_1X, BB_1X to intersect in O. Then, the triangles BOA, B_1OA_1, being equiangular, are similar; and O is similarly situated in respect to the two figures ABCDE, $A_1B_1C_1D_1E_1$;

hence, $\angle BOC = B_1O_1C_1$;

therefore $COC_1 = BOB_1 = AOA_1$, &c.



2. Divide AA_1 in M, so that

$$AM : A_1M = AB : A_1B;$$

and through M draw MN, equally inclined to AB and A_1B_1. Then MN is evidently symmetrically placed as regards the two polygons. Draw AO, A_1O to meet in MN at equal angles with it; then the angle $OAB = OA_1B_1$; and O is symmetrically situated with regard to the two polygons, therefore ON bisects BOB_1, COC_1, &c.

[In (1) the angle which any two homologous points subtend is the angle between any two homologous sides of the polygon; that is to say, it is the angle through which either polygon must be turned to bring the two into corresponding positions. From the above it is clear that two similar curves can always be expressed by one of the systems of equations

$$r = af(\theta), \quad \rho = bf(\theta - \alpha), \quad \text{or} \quad r = af(\theta), \quad \rho = bf(\alpha - \theta)].$$

6111. (By A. JOHNSON, M.A.)—A raft, whose length, breadth, and depth are l, b, d feet respectively, is made of wood, whose specific gravity is s; find the utmost weight it will bear before sinking.

Solution by E. W. SYMONS, B.A.; E. RUTTER; *and others.*

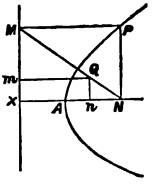
Supposing the raft to be just totally immersed when a weight W is placed on it (without altering shape and volume of raft), we have, equating weight of raft to fluid pressure,

$$glbd \cdot s + W = glbd, \text{ therefore } W = glbd\,(1-s).$$

6180. (By R. E. RILEY, B.A.)—From any point P of a parabola PM, PN are drawn perpendicular to the directrix and axis. If Q be a point in MN, dividing it in any constant ratio, prove that the locus of Q is a co-axal parabola.

Solution by R. KNOWLES, B.A., L.C.P.; D. EDWARDES; *and others.*

Let MQ = m . QN, and let Q be (X, Y); then we have $\frac{m}{1+m} = \frac{X}{x}$, $1+m = \frac{y}{Y}$,



the origin being the foot (X) of the directrix;
but $y^2 = 4a(x-a)$,
therefore $(1+m)^2 Y^2 = 4a\{(1+m^{-1})X - a\}$,
which proves the theorem.

2756. (By J. GRIFFITHS, M.A.)—Show that an infinite number of triangles can be described such that each has the same circumscribed nine-point, and self-conjugate circles as a given triangle.

Solution by Professor WOLSTENHOLME, M.A.

Let ABC be the given triangle, P any point on the circumscribed circle, and let the polar of P, with respect to the self-conjugate circle, meet the circumscribed in QR; then PQR will also be self-conjugate; for, if not, let R′ be the pole of PQ, then R′ must lie on QR, the polar of P, and the triangle PQR′ is self-conjugate to the circle. But ABC is also; therefore the six points A, B, C, P, Q, R′ lie on one conic, and five of them are on the circumscribed circle; therefore R′ is also on that circle, and coincides with R.

If the circumscribed circle and self-conjugate circle of a triangle be given, so also is the nine-point circle, since its centre bisects the distance between the two given centres, and its radius is half one of the given radii.

The sides will obviously all touch the hyperbola, of which the two given centres are foci, whose major axis is equal to the radius of the given circumscribed circle, and the nine-points circle will be its auxiliary circle.

[A solution by Prof. CAYLEY is given on p. 98 of Vol. X. of the *Reprint*.]

6045. (By R. F. DAVIS, M.A.)—APQ is a triangle similar to a given fixed triangle. If A be fixed, and the locus of P a straight line, show that the locus of Q is also a straight line.

Solution by T. MORLEY, L.C.P.; H. MURPHY; *and others.*

This theorem is obvious. We have two radii AP, AQ whose ratio is given, and the angle between them. If, then, P describe any locus, Q will

describe a similar locus, turned through an angle PAQ. If (P) be a conic given by the general equation, and $\angle PAQ = \alpha$, and $AP = m \cdot AQ$, the conic (Q) is easily seen to be

$$m^2x^2 (a\cos^2\alpha + b\sin^2\alpha + h\sin 2\alpha)\ 2y$$
$$+ m^2y^2 (a\sin^2\alpha + b\cos^2\alpha - h\sin 2\alpha) + 2m^2xy [\tfrac{1}{2}(b-a)\sin 2\alpha + h\cos 2\alpha]$$
$$+ 2mx (g\cos\alpha + f\sin\alpha) + 2my (f\cos\alpha - g\sin\alpha) + c = 0.$$

[By reference to the Editorial Note on the Solution of Quest. 4149, on p. 49 of Vol. XXII. of the *Reprint*, the same general result will be seen to follow at once. Mr. DAVIS states that the question was suggested by 683 in the 2nd edition of Prof. WOLSTENHOLME's collection of Problems.]

5355. (By S. M. DRACH, F.R.A.S.)—

Hindoo chess-inventors' claim bred despair;
'Twas doubling wheat-grains on following square.
How long 'twould feed mankind I here declare.
Four hundred millions' daily quartern loaf,
Ta'en from this fund, I find would be enough
From Noah's deluge human folk to sate,
Past one-nine-eighteen of the Christian date.
If useless Bacco-Bacchus men don't inhale,
Good corn-bread would suffice up to this tale.

Solution by the PROPOSER.

Grains $2^{64}-1$; in a quarter $512\times256\times30$; quartern loaves $577\times4\frac{11}{18}$, Mackenzie's Tab., p. 16; result, $(2^{52}\times577)$ divided by $(1461\times417\times10^8)$, or $4265\frac{1}{4}$ years, or $2347+1918$.

5998. (By C. LEUDESDORF, M.A.)—A lamina in the form of an ellipse has one point P in its circumference fixed, and is struck perpendicularly to its plane at another point Q in the circumference; show that, if it begin to rotate about the tangent to the ellipse at P, then PQ is a diameter.

Solution by T. R. TERRY, M.A.; Prof. MATZ, M.A.; *and others.*

Taking the lines through P parallel to the axes as axes of x and y, let ω_x, ω_y be the angular velocities round these axes generated by the blow B; then, taking moments round these axes, and putting M the mass of the lamina, we have

$$M(\tfrac{1}{4}b^2 + b^2\sin^2\theta)\,\omega_x - Mab\sin\theta\cos\theta\,\omega_y = Bb(\sin\phi - \sin\theta) \quad \ldots\ldots(1),$$
$$M(\tfrac{1}{4}a^2 + a^2\cos^2\theta)\,\omega_y - Mab\sin\theta\cos\theta\,\omega_x = -Ba(\cos\phi - \cos\theta) \quad \ldots(2).$$

If the body begins to turn round the tangent at P, we have

$$\frac{\omega_x \cos\theta}{a} + \frac{\omega_y \sin\theta}{b} = 0, \text{ or } \frac{b\omega_x}{a\omega_y} = \frac{\sin\theta}{-\cos\theta};$$

therefore, from (1) and (2), by division, we obtain

$$\frac{(\frac{1}{3} + \sin^2\theta)\sin\theta + \sin\theta\cos^2\theta}{-(\frac{1}{3} + \cos^2\theta)\cos\theta - \sin^2\theta\cos\theta} = \frac{\cos\frac{1}{2}(\theta+\phi)}{\sin\frac{1}{2}(\theta+\phi)};$$

therefore $\sin\theta\sin\frac{1}{2}(\theta+\phi) + \cos\theta\cos\frac{1}{2}(\theta+\phi) = 0$, or $\theta - \phi = \pi$;

hence PQ is a diameter of the ellipse.

5815. (By S. Tebay, B.A.)—Property worth £400 is mortgaged for s pounds, and afterwards sold out for an integral number (t) of pounds, which is s per cent. on the mortgage. If $s+t$ be $>$ £229, but $<$ £335, find the probable amount realized by the transaction.

Solution by the Proposer.

First, $\frac{t}{s} = \frac{s}{100}$; or $s = 10\,(t)^{\frac{1}{2}}$. Let $u_t = 10\,(t)^{\frac{1}{2}} + t$; then

$$\Sigma u_{t+1} = \int u_{t+1}\,dt - \tfrac{1}{2}u_{t+1} + \frac{B_1}{2!}\frac{du_{t+1}}{dt} - \frac{B_3}{4!}\frac{d^3u_{t+1}}{dt^3} + \&c. + C$$

(B_1, B_3, &c. being Bernoulli's numbers)

$$= \tfrac{20}{3}(t+1)^{\frac{3}{2}} + \tfrac{1}{2}(t+1)^2 - \tfrac{1}{2}[10\,(t+1)^{\frac{1}{2}} + t + 1] + \tfrac{1}{12}[5\,(t+1)^{-\frac{1}{2}} + 1] - \tfrac{1}{168}(t+1)^{-\frac{3}{2}} + \&c. + C.$$

The limits of t are 120 and 195; therefore, the average required is

$$\tfrac{1}{75}\Sigma_{t=120}^{t=195} u_{t+1} = \tfrac{1}{75}[\tfrac{20}{3}(14^3 - 11^3) + \tfrac{1}{2}(196^2 - 121^2) - \tfrac{1}{2}(140 + 196 - 110 - 121) + \tfrac{1}{12}(\tfrac{5}{14} + 1 - \tfrac{5}{11} - 1)] = £283.\ 5s.\ 4d.$$

5977. (By Edwyn Anthony, M.A.)—If θ, θ' be the eccentric angles corresponding to the extremities of the chord PQ of an ellipse, which is normal at P, $\tan\theta\tan\frac{1}{2}(\theta+\theta') = e^2 - 1$.

Solution by R. Graham, M.A.; A. W. Scott, M.A.; *and others.*

The equations of PQ and the tangent at P are

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\theta') + \frac{y}{b}\sin\tfrac{1}{2}(\theta+\theta') = \cos\tfrac{1}{2}(\theta-\theta'),\ \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1;$$

and, since PQ is a normal, these lines are at right angles;

therefore $$\frac{\cos\theta\cos\frac{1}{2}(\theta+\theta')}{a^2}+\frac{\sin\theta\sin\frac{1}{2}(\theta+\theta')}{b^2}=0,$$

or $$\tan\theta\tan\tfrac{1}{2}(\theta+\theta')=-\frac{b^2}{a^2}=e^2-1.$$

6168. (By the Editor.)—A coin is dropped at random over an indefinite horizontal grating formed of parallel equidistant wires; prove that, according as the distance between each pair of wires is equal to (1) the circumference, (2) the diameter, (3) the radius, of the coin, the respective chances that the coin will strike in passing through the grating are

$$p_1=\tfrac{1}{4},\quad p_2=\tfrac{1}{4}\pi,\quad p_3=\tfrac{1}{8}\pi+\tfrac{1}{4}\sqrt{3}.$$

I. *Solution by* C. J. Monro, M.A.

Neglect the thickness of the coin and wire, and, in comparison with its velocity of translation, the coin's velocity of rotation. Let a represent the distance between adjacent wires in magnitude or direction, and let h be the coin's breadth parallel to a; then the chance, for a particular value of h, is $\frac{h}{a}$, when $h<a$, and 1 when $h>a$.

Let c be the diameter of the coin, and α the inclination of its normals to a, then $h=c\sin\alpha$. Therefore, if $c\sin A=a$, and $Q\,d\alpha$ be the chance that the inclination is between α and $\alpha+d\alpha$, the chance required (call it p) is $\frac{c}{a}\int_0^{\frac{1}{2}\pi}Q\sin\alpha\,d\alpha$, if $a>c$; and $\frac{c}{a}\int_0^{A}Q\sin\alpha\,d\alpha+\int_A^{\frac{1}{2}\pi}Q\,d\alpha$, if $a<c$.

As the coin is dropped at random, its plane is disposed at random, and its normals directed at random. When, in such cases, all depends on inclination to a plane fixed in disposition (as here, a plane perpendicular to a), there is a tendency to distribute the inclinations uniformly. See Cournot's *Exposition de la Théorie des Chances*, p. 270, on the inclinations of the planes of the cometary orbits. But random distribution requires geometrical invariance, or independence of all fixed loci; and it seems to follow that the solid angle of a cone, divided by its integral, is the measure of the chance that a direction taken at random shall fall within the range bounded by such cone. Then $Q=\sin\alpha$, and p is

$$\frac{\pi}{4}\cdot\frac{c}{a},\ \text{or}\ \frac{\pi}{4}\sin A,\quad \text{if } a>c;$$

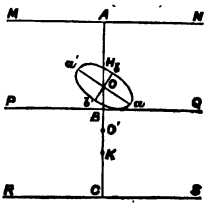
and $$\frac{1}{2a}\left\{c\sin^{-1}\frac{a}{c}+(c^2-a^2)^{\frac{1}{2}}\right\},\ \text{or}\ \frac{1}{2}\left(\frac{A}{\sin A}+\cos A\right),\ \text{if } a<c.$$

These formulas will give the Proposer's results.

II. *Solution by* Professor Seitz, M.A.

Let MN, PQ, RS represent three consecutive wires of the grating; ABC a perpendicular intersecting them in A, B, C; H and K the middle points of AB and BC.

It is evidently necessary to consider only those cases in which the centre of the coin in a free fall would pass through some point of HK. Let the ellipse $aba'b'$ be the projection of the circumference of the coin on the plane of the grating when it just touches the wire PQ in passing through the grating, aa' and bb' being the axes of the ellipse, and the centre O being in HK. Take BO′ equal to BO. Let θ = the angle with which the coin meets the grating, r = the radius of the coin, OO′ = x, and $\angle BOb' = \phi$.



Then we have $Oa = r$, $Ob = r\cos\theta$, and since OB is the perpendicular from the centre on the tangent PQ, $x = 2r\,(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}$.

1. Let HK = $2\pi r$; then we have

$$p_1 = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} x\,d\phi\sin\theta\,d\theta + \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} 2\pi r\,d\phi\sin\theta\,d\theta$$

$$= \frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}\,d\phi\sin\theta\,d\theta$$

$$= \frac{1}{\pi^2}\int_0^{\frac{1}{2}\pi}\left\{1+\cos\phi\log\tan\tfrac{1}{2}\phi-\sec\phi\log\tan\tfrac{1}{2}\phi\right\}d\phi$$

$$= \frac{1}{\pi^2}\left\{\phi+\sin\phi\log\tan\tfrac{1}{2}\phi-\phi\right\}_0^{\frac{1}{2}\pi} - \frac{1}{\pi^2}\int_0^{\frac{1}{2}\pi}\sec\phi\log\tan\tfrac{1}{2}\phi\,d\phi$$

$$= \frac{1}{\pi^2}\int_0^{\frac{1}{2}\pi}(1+\tfrac{1}{2}\cos^2\phi+\tfrac{1}{3}\cos^4\phi+\tfrac{1}{4}\cos^6\phi+\ldots)\,d\phi$$

$$= \frac{1}{\pi^2}\cdot\frac{\pi}{2}\left(1+\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{3}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}+\tfrac{1}{4}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{1}{2}+\ldots\right) = \frac{1}{\pi^2}\cdot\frac{\pi^2}{4} = \frac{1}{4}.$$

2. Let HK = $2r$; then we have $p_2 = \pi p_1 = \frac{1}{4}\pi$.

3. Let HK = r; then, if $\phi < \frac{1}{6}\pi$ and $\theta > \sin^{-1}(\frac{1}{3}\sqrt{3}\,.\sec\phi)$, or θ_1, the number of ways the coin can fall so as to strike in passing, is x; but if $\phi < \frac{1}{6}\pi$ and $\theta < \theta_1$, or $\phi > \frac{1}{6}\pi$ and θ has any value, the number of ways is r. Hence we have

$$p_3 = \frac{\displaystyle\int_0^{\frac{1}{6}\pi}\left(\int_0^{\theta_1} r\sin\theta\,d\theta+\int_{\theta_1}^{\frac{1}{2}\pi} x\sin\theta\,d\theta\right)d\phi+\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} r\,d\phi\sin\theta\,d\theta}{\displaystyle\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} r\,d\phi\sin\theta\,d\theta}$$

$$= \frac{2}{\pi}\int_0^{\frac{1}{6}\pi}\left\{\int_0^{\theta_1}\sin\theta\,d\theta+\int_{\theta_1}^{\frac{1}{2}\pi}2(1-\sin^2\theta\cos^2\phi)^{\frac{1}{2}}\sin\theta\,d\theta\right\}d\phi+\frac{2}{\pi}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} d\phi\sin\theta\,d\theta$$

$$= \frac{2}{3}+\frac{1}{\pi}\int_0^{\frac{1}{6}\pi}\left\{2-(1-\tfrac{3}{4}\sec^2\phi)^{\frac{1}{2}}+2(\cos\phi-\sec\phi)\log\left[\frac{2\sin\phi}{1+(1-4\sin^2\phi)^{\frac{1}{2}}}\right]\right\}d\phi$$

$$= \frac{\sqrt{3}}{4}-\frac{2}{\pi}\int_0^{\frac{1}{6}\pi}\sec\phi\,d\phi\log\left[\frac{2\sin\phi}{1+(1-4\sin^2\phi)^{\frac{1}{2}}}\right].$$

Let $2\sin\phi = \sin\psi$; then we have

$$p_3 = \tfrac{1}{4}\sqrt{3} - \frac{8}{\pi}\int_0^{\frac{1}{6}\pi} \frac{\cos\psi}{4-\sin^2\psi} \log\tan\tfrac{1}{2}\psi\, d\psi$$

$$= \tfrac{1}{4}\sqrt{3} + \frac{2}{\pi}\left\{\log\left(\frac{2-\sin\psi}{2+\sin\psi}\right)\log\tan\tfrac{1}{2}\psi\right\}_0^{\frac{1}{6}\pi} - \frac{2}{\pi}\int_0^{\frac{1}{6}\pi} \operatorname{cosec}\psi\, d\psi \log\left(\frac{2-\sin\psi}{2+\sin\psi}\right)$$

$$= \tfrac{1}{4}\sqrt{3} + \frac{2}{\pi}\int_0^{\frac{1}{6}\pi}\left(\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2^2}\sin^2\psi + \frac{1}{5}\cdot\frac{1}{2^5}\sin^4\psi + \frac{1}{7}\cdot\frac{1}{2^7}\sin^6\psi + \ldots\right)d\psi$$

$$= \tfrac{1}{4}\sqrt{3} + \left(\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2^2} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}\cdot\frac{1}{2^5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7}\cdot\frac{1}{2^7} + \ldots\right)$$

$$= \tfrac{1}{6}\pi + \tfrac{1}{4}\sqrt{3}.$$

[A *general* solution by the Proposer is given on pp. 56–58 of Vol. I. of the *Reprint*.]

1433. (By the Editor.)—Prove the following reciprocal cases of involution:—

a. The three sides of every triangle, and every three concurrent lines through its three vertices, intersect every axis in six points in involution.

a'. The three vertices of every triangle, and every three collinear points on its three sides, subtend every vertex in six rays in involution.

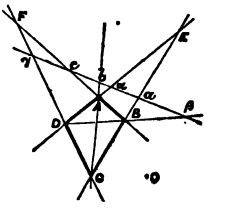
b. The six perpendiculars on the six lines from any point in the former case determine at the point a pencil of six rays in involution.

b'. The six perpendiculars from the six points upon any line in the latter case determine on the line a system of six points in involution.

Solution by the late Professor Clifford, F.R.S.

Six points are in involution when the anharmonic ratio of any four is equal to that of their four conjugates.

a. Let then ABC be the triangle, D the point of concourse; and let a straight line meet the sides and corresponding lines in $a, b, c, \alpha, \beta, \gamma$ respectively; then we have



$$[abc\gamma] = \{A\,.\,DCB\gamma\} = \{A\,.\,DCF\gamma\}$$
$$= \{B\,.\,DCF\gamma\} = [\beta ac\gamma] = [a\beta\gamma c],$$

which proves the proposition.

a'. Let now ADF be the triangle, C, B, E the collinear points, and take any point O; then we have

$$\{O\,.\,ABDE\} = \{B\,.\,AODE\} = \{B\,.\,FODC\}$$
$$= \{O\,.\,FBDC\} = \{O\,.\,CDBF\},$$

which proves the second case.

b. The line at infinity is cut in involution, by prop. (*a*); hence, lines

parallel to the given six through any point will form a pencil in involution; turn this pencil through a right angle, and it coincides with the perpendiculars.

b'. Any point at infinity is subtended in involution, by (a'); whence the theorem immediately follows.

5868. (By S. Tebay, M.A.)—Two vessels contain mixtures of wine and water; the first m gallons of wine, and a gallons of water; the second n gallons of wine, and b gallons of water; find the probability that a mixture formed at hazard will contain at least an average of the whole wine.

Solution by the Proposer.

Let the mixture consist of x gallons from the first vessel, and y gallons from the second; these contain $\frac{m}{m+a}x$ and $\frac{n}{n+b}y$ gallons of wine respectively; hence, putting $\frac{m}{m+a} = \mathrm{M}$, and $\frac{n}{n+b} = \mathrm{N}$, we must have

$$\mathrm{M}x + \mathrm{N}y > \tfrac{1}{2}(m+n).$$

Let $m > n$; then $\frac{1}{2}(m+n) > n$; therefore the limits of y are 0 and $n+b$. For any value of y, the range of x

$$= m+a-\frac{1}{\mathrm{M}}\left\{\tfrac{1}{2}(m+n)-\mathrm{N}y\right\} = \frac{m-n+2\mathrm{N}y}{2\mathrm{M}};$$

therefore, chance $= \int_0^{n+b}(m-n+2\mathrm{N}y)\,dy + 2\mathrm{M}\,(m+a)\,(n+b) = \tfrac{1}{2}$.

This result is pretty evident without calculation; for the chance that the mixture contains not less than $\frac{1}{2}(m+n)$ wine must be equal to the chance that it contains not more than $\frac{1}{2}(m+n)$ wine, that is, $\frac{1}{2}$.

5845. (By W. H. Lowry, M.A.)—Show that the equation whose roots are the cubes of the roots of $f(x) \equiv x^n + \mu x^{n-1} + \ldots = 0$, may be written

$$f(y^{\frac{1}{3}})\left\{(\phi_1 y)^2 - \phi_2(y)\cdot\phi_3(y)\right\} = 0;$$

and determine $\phi_1(y)$, $\phi_2(y)$, $\phi_3(y)$.

Solution by Prof. Scott, M.A.; *the* Proposer; *and others.*

If ω and ω^2 be the imaginary cube roots by unity, we have

$$f(u)\cdot f(\omega x)\cdot f(\omega^2 u) \equiv (x^3-\alpha^3)(x^3-\beta^3)\ldots,$$

that is, $$Y_1^3+Y_2^3+Y_3^3-3Y_1Y_2Y_3,$$

or $$(Y_1+Y_2+Y_3)\left\{(Y_1-Y_3)^2-(Y_3-Y_1)(Y_3-Y_2)\right\} \equiv (y-\alpha^3)(y-\beta^3)\ldots\ldots,$$

where $$Y_1 = y^{\frac{1}{3}n}+p_3 y^{\frac{1}{3}(n-3)}+p_6 y^{\frac{1}{3}(n-6)}\ldots\ldots,$$

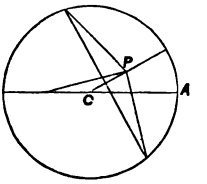
$$Y_2 = p_1 y^{\frac{1}{3}(n-1)}+p_7 y^{\frac{1}{3}(n-4)}+p_7 y^{\frac{1}{3}(n-7)},$$

$$Y_3 = p_2 y^{\frac{1}{3}(n-2)}+p_5 y^{\frac{1}{3}(n-5)}+p_8 y^{\frac{1}{3}(n-8)};$$

therefore $\phi_1(y) = Y_1-Y_3$, &c.

5167. (By Prof. Seitz, M.A.)—A point is taken at random in the surface of a given circle, and from it a line equal in length to the radius is drawn, so as to lie wholly in the surface of the circle; find the chance that the line intersects a given diameter.

Solution by Rev. H. J. Day, M.A.; E. B. Elliott, M.A.; *and others.*



Let ρ be the distance of the random point P from the centre of the circle C; c the radius of the circle; and ϕ the inclination of CP to the given diameter.

Then the whole range of the angles at which the radius can be drawn as prescribed through p, is $2\cos^{-1}\frac{\rho}{2c}$, and the number that cut the diameter in that set is, evidently,

$$\cos^{-1}\frac{\rho}{2c}+\phi-\sin^{-1}\left(\frac{\rho}{c}\sin\phi\right);$$

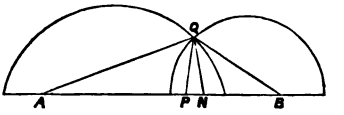
hence the required probability (where $\rho<c$, and $\phi<\frac{1}{2}\pi$)

$$\iint \frac{\phi+\cos^{-1}\left(\frac{\rho}{2c}\right)-\sin^{-1}\left(\frac{\rho\sin\phi}{c}\right)}{2\cos^{-1}\frac{\rho}{2c}}\cdot\frac{4\rho\,d\rho\,d\phi}{c^2\pi}.$$

[This expression does not seem to admit of integration in finite terms.]

5758. (By Rev. H. G. Day, M.A.)—If A, B be two fixed points, and P a variable point in the line AB, and if with A, P as foci an ellipse of eccentricity e_1 be drawn, and with B, P as foci an ellipse of eccentricity e_2, intersecting the former in Q; prove that the locus of Q is an ellipse with A, B as foci.

Solution by Professor NASH, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*



Let AB = $2c$,

AP = 2ρ, PN = x;

then AQ = $\rho e_1^{-1} + e_1(\rho + x)$,

PQ = $\rho e_1^{-1} - e_1(\rho + x)$,

BQ = $(c-\rho)e_2^{-1} + e_2(c-\rho-x)$,

PQ = $(c-\rho)e_2^{-1} - e_2(c-\rho-x)$;

therefore $x = \frac{\rho(1-e_1^2)}{e_1(e_1+e_2)} - \frac{(c-\rho)(1-e_2^2)}{e_2(e_1+e_2)}$; hence

$$AQ + BQ = \frac{\rho}{e_1}(1+e_1^2) + \frac{c-\rho}{e_1+e_2}(1+e_2^2) + (e_1-e_2)x$$

$$= \frac{\rho}{e_1+e_2}(2+2e_1e_2) + \frac{c-\rho}{e_1+e_2}(2+2e_1e_2) = \frac{2c(1+e_1e_2)}{e_1+e_2} = \text{a const.}$$

4817. (By Professor SYLVESTER, F.R.S.)—In a school of 15 girls, a rule has been laid down that they shall walk out every day in rows of threes, but that the same two girls shall never come together twice in the same row. The rule is supposed to have been carried out correctly during the six working days of the week, but when the time comes for their going to church together on Sunday it is found to be absolutely impossible to continue it any further. Investigate whether the rule *can* have been correctly carried out during the six previous days; and if so, show how.

Solution by the PROPOSER.

The question *is* possible, and may be answered in various ways.

One solution will leave out the duads 1 . 2, 1 . 3, 2 . 3, 4 . 5, 4 . 6, 5 . 6, 7 . 8, 7 . 9, 8 . 9, 10 . 11, 11 . 12, 12 . 13, 13 . 14, 14 . 15, 15 . 10, which 15 duads evidently cannot be contained in 5 triads, on account of the *cycle*

10, 11, 12, 13, 14, 15.

5403. (By Dr. HOPKINSON, F.R.S.)—An iron hoop, 10 feet in diameter, revolves about an axis through its centre, perpendicular to its plane, with a velocity ω; find the greatest value ω can have without breaking the hoop.

Solution by J. J. WALKER, M.A.

Let T be the tensile strength, w the weight in lbs. weight per cubic inch, R the radius, a the breadth, and b the (small) thickness, of the hoop;

then the critical value of ω is determined by

$$2\pi R\, wab\, \omega^2 R = g\, 2\pi T\, ab, \quad \text{or} \quad \omega^2 = gT + R^2 w,$$

where g and R are expressed in inches. Taking the tensile strength to be 10 tons weight to the square inch, this would give, for $R = 60$, $\omega = 90$ about; say 14 revolutions per second.

Multiplying the equation above by dR, integrating and reducing, we have the more general result $\omega^2 = gT + R'R''w$ for a fly-wheel of exterior and interior radii R', R'' respectively.

6215. (By Dr. Francesco Tirelli.)—Siano OA, OB, OC tre corde di un cerchio, sulle quali come diametri si descrivano tre circonferenze C_1, C_2, C_3. Indichiamo con A', B', C' le ulteriori intersezioni di C_2 con C_3, C_3 con C_1, C_1 con C_2. Inoltre C_1 incontri OB ed OC in β e γ, C_2 incontri OC ed OA in γ_1 ed α_1 e C_3 incontri OA ed OB in α_2 e β_2. Dimostrare, che i punti A', B', C' stanno per dritto, e che le rette $\beta\gamma$, $\gamma_1\alpha_1$, $\alpha_2\beta_2$ concorrono in uno stesso punto della A'B'C'.

Solution by the Proposer.

Si conducano OA', OB', OC', AC', C'B, BA', A'C, CB', B'A. E facile vedere, che A', B', C' sono i piedi delle perpendicolari condotte da O su' lati del triangolo ABC, perciò stanno per dritto.

Poichè $C'\widehat{A}B' = B\widehat{O}\gamma$, segue $B'C' = \beta\gamma$. E se indichiamo con A'', B'', C'' i piedi delle perpendicolari condotte da' vertici del triangolo ABC su' lati opposti, segue, come si è dedotto $B'C' = \beta\gamma$, che deduconsi $C'A' = \gamma_1\alpha_1$, $A'B' = \alpha_2\beta_2$, ..., cioè che le quattro rette $A''\beta\gamma$, $B''\gamma_1\alpha_1$, $C''\alpha_2\beta_2$, A'B'C' sono tali, che i segmenti compresi fra i tre punti segnati su ciascuna sono uguali. E se X è il punto comune ad $A''\beta\gamma$, A'B'C', si ha

$$XB' \cdot XC' = X\gamma \cdot X\beta;$$

da cui segue pel'osservazione testè fatta $XB' = X\beta$, $XC' = X\gamma$, e quindi, per essere $A''\gamma = A'C'$, si ha $XA'' = XA'$. Adunque le quattro rette $A''\beta\gamma$, $B''\gamma_1\alpha_1$, $C''\alpha_2\beta_2$, A'B'C' sono tali, che i segmenti fra due punti di ciascuna sono uguali, e due di esse si incontrano in un punto che serba uguali distanze da' punti dell' una e dell' altra.

Ciò posto; se ABC, A'B'C', $A_1'B_1'C_1'$ sono tre rette che soddisfanno alle condizioni testè cennate, e che si tagliano due a due in O, O_1, O_2, si ha $OA = OA'$, $O_1A = O_1A_1'$, $O_2A' = O_2A_1'$, da cui agevolmente segue $OO_2 - O_1O_2 = OO_1$ che è assurda. Perciò le quattro rette $A''\beta\gamma$, $B''\gamma_1\alpha_1$, $C''\alpha_2\beta_2$, A'B'C' concorrono in un punto.

[See the Solutions of Quest. 2444, on p. 48 of Vol. VIII. of the *Reprint*.]

5992. (By Elizabeth Blackwood.)—If P, Q, R, S be four random points on the surface of a given sphere; find the chance that the point S will be within the spherical triangle PQR.

Solution by Professor SEITZ, M.A.

The required chance is evidently equal to the average area of the triangle PQR, divided by the area of the surface of the sphere. By the solution of Quest. 4849 (*Reprint*, Vol. XXV., p. 54), the average area of the triangle PQR is equal to the area of a tri-rectangular triangle. Hence the required chance is $\frac{1}{8}$.

6098. (By Prof. COCHEZ.)—Trouver dans le plan de trois cercles un point tel que les tangentes menées de ce point aux troix cercles soient entre elles comme trois droites données.

Solution by J. A. KEALY, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Let $f(x, y) = 0$ and $\phi(x, y)$ be two of the circles; then $f(x, y) = k \cdot \phi(x, y)$ is the locus of a point from which the tangents to the circles are in a given ratio, and this locus must evidently be some circle.

Hence we have the following geometrical construction:—Draw the four common tangents, or at least three of them, to two of the circles; divide them in the given ratio, and draw a circle through the points of division; do the same with another pair; then the intersections of these two circles will give the points required.

5538. (By Professor TOWNSEND, F.R.S.)—A heavy flexible chain, of uniform thickness, being supposed to occupy longitudinally just half the interior of a slender circular tube, of uniform roughness, bounded radially by coaxal horizontal cylinders, and laterally by parallel vertical planes; determine, given all particulars, its two extreme positions of equilibrium under the action of gravity.

Solution by the PROPOSER.

Denoting by a the radius of the tube, by m the mass per unit of length of the chain, by θ the angle between the radius at any point P of its length and the horizontal radius of the tube through whose extremity it does not pass, by T and R the tension of the chain and the reaction of the tube at P, and by μ and ϵ the coefficient and angle of the friction arising from the uniform roughness of the tube; then, since, by tangential and normal resolution, in the usual manner,

$$\frac{d\mathrm{T}}{d\theta} - \mu a\mathrm{R} = amg\cos\theta, \text{ and } \mathrm{T} - a\mathrm{R} = -am\sin\theta;$$

and since, by eliminating R, $\frac{d\mathrm{T}}{d\theta} - \mu\mathrm{T} = amg(\mu\sin\theta + \cos\theta)$;

therefore, by integration, we have in finite terms,

$$T = C\epsilon^{\mu\theta} + \frac{amg}{1+\mu^2}\left[(1-\mu^2)\sin\theta - 2\mu\cos\theta\right],$$

the value of the arbitrary constant C depending, of course, on the particulars of the case.

But, since, in the case in question, $T=0$ when $\theta=\theta_0$ and $=\theta_0+\pi$, where θ_0 = the initial value of θ; therefore, whatever be the value of μ, $C=0$; consequently, for the initial and terminal values of θ,

$$\tan\theta_0 = \tan(\theta_0+\pi) = \frac{2\mu}{1-\mu^2} = \tan 2\epsilon;$$

therefore $\theta_0 = 2\epsilon$; and therefore, &c.

6125. (By Professor Monck, M.A.)—Find examples of right-angled triangles, with commensurable sides, such that (1) one, (2) neither of the acute angles is less than $\sin^{-1}\frac{3}{5}$.

Solution by G. Heppel, M.A.; Rev. W. A. Whitworth, M.A.; *and others.*

1. The following are examples of such right-angled triangles:—

15 : 8 : 17, 35 : 12 : 37, 63 : 16 . 65, 99 : 20 : 101, 39 : 80 : 89,
5 : 12 : 13, 7 : 24 : 25, 9 : 40 : 41, 11 : 60 : 61, 45 : 28 : 53,
33 : 56 : 65, 13 : 84 : 85, 77 : 36 : 85.

2. If the hypotenuse be less than 100, the following are the only triangles in which neither of the acute angles is less than $\sin^{-1}\frac{3}{5}$:—

3 : 4 : 5, 21 . 20 : 29, 65 : 72 : 97, 55 : 48 : 73.

6146. (By the Rev. W. Roberts, M.A.)—Two tangents to a given parabola make angles with the axis such that the product of the tangents of their halves is constant; prove that the locus of the intersection of the tangents is a confocal parabola.

Solution by E. W. Symons, B.A.; R. Knowles, L.C.P.; *and others.*

The relation $\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta = k$ is equivalent to

$$4k^2(\tan^2\alpha+\tan^2\beta) + 4k(1+k^2)\tan\alpha\tan\beta = (1-k^2)^2\tan^2\alpha\tan^2\beta,$$

and if $\tan\alpha$, $\tan\beta$ be the roots of the quadratic in m,

$$y = m(x+a) + \frac{a}{m} \quad \text{or} \quad m^2(x+a) - my + a = 0,$$

we get $4k^2\left\{y^2 - 2a(x+a)\right\} + 4k(1+k^2)a(x+a) = (1-k^2)^2a^2$

or $$y^2 = -\frac{(1-k)^2}{k}a\left(x - \frac{(1-k)^2}{4k}a\right),$$

a parabola confocal with the original. It is obvious that we should get the same locus from the relation

$$\tan \tfrac{1}{2}\alpha = -k \tan \tfrac{1}{2}\beta \quad \text{or} \quad \tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta = k^{-1}.$$

When $k = -1$, $(\tan \alpha - \tan \beta)^2 = 0$, the two tangents must coincide, and the locus reduces to the original $y^2 = 4a(x+a)$.

5917. (By W. H. H. Hudson, M.A.)—If the capital that is invested in machinery and raw material be proportional to the rate of interest, and if the proceeds of labourers' work after replacing capital be proportional to their wages and to the aforesaid capital conjointly, and if the employers, after paying wages and interest on the aforesaid capital, make the greatest possible profit; prove that the rate of interest in the pound must be equal to the ratio of the proceeds above mentioned to twice the aforesaid capital.

Solution by W. J. Macdonald, M.A.; Christine Ladd; *and others.*

Let C = capital, R = rate of interest per £1, I = interest on capital, P = proceeds, W = wages, G = profit; then we have

$$C = mR, \quad P = nWC, \quad G = P-(W+I), \quad I = CR = mR^2 + P = mnWR.$$

Therefore $G = mnWR - W - mR^2 = \text{a max.}$; $\therefore \frac{dG}{dR} = mnW - 2mR = 0$;

therefore $nW = 2R$; therefore $R = \frac{nWC}{2C} = \frac{P}{2C}$.

6061. (By J. E. A. Steggall, M.A.)—If n letters are put at random into their n directed envelopes, find (1) the chance that no letter goes right; and (2) show that when n is infinite the chance is e^{-1}.

Solution by the Rev. W. Allen Whitworth, M.A.

The number of ways in which the letters can be put into the envelopes is $n!$ The number of ways so that all may be wrong is (*Choice and Chance*, Prop. xxxii.) $n!!$ Therefore the chance is

$$\frac{n!!}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots \pm \frac{1}{n!}.$$ If $n = \infty$ the result becomes $\frac{1}{e}$.

In *Choice and Chance*, 3rd Edition, p. 234, Ex. 233, I have pointed out that, if the number of letters exceed 8, the chance of all going wrong is

(correct to six decimal places) ·367879. Hence the chance of all going wrong is not *appreciably* affected by the number of letters. Whether there are 10 letters or 100 or 1000, the chance only differs from e^{-1} by a decimal figure of a very high order.

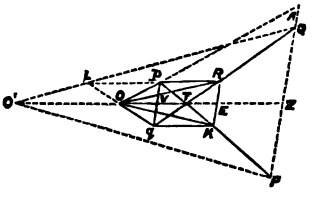
[The discussion of this famous historical problem began, so far as we know, with MONTMORT, in 1708, under the name of the game of *Treize*, and it has since been continued and extended by DE MOIVRE, EULER, LAPLACE, and others. Historical references are supplied in an editorial foot-note to Quest. 2397 (*Reprint*, Vol. VIII., p. 55), where solutions to another form of the Question are given by Messrs. WOOLHOUSE and WHITWORTH.

To suit our Printer's convenience, we use, here and everywhere, the symbol $n!$ for factorial n, and we have, in the foregoing solution, put $n!!$ for what Mr. WHITWORTH calls sub-factorial n, for the definition whereof we refer to p. 98 of *Choice and Chance*.]

6108 & 6109. (By ELIZABETH BLACKWOOD and L. H. ROSENTHAL, B.A.)—P, Q, p, q are four points on a parabola, such that Pp and Qq are respectively parallel to the tangents at q and p. If O and O′ are the poles of pq and PQ respectively, and if O_{PQ} represents the perpendicular from O on PQ, &c., prove that (1) PQ = $3pq$, (2) $O_{PQ} = O'_{pq}$, (3) $O'_{PQ} = 9O'_{pq}$, (4) $O_{PQ} = 5O_{pq}$; (5) find the ratio of the areas of the triangles Opq, O′PQ; (6) find the locus of the intersections with PQ of tangents at p, q, (α) when pq passes through the focus, and (β) when pq is parallel to a fixed line.

Solutions by D. EDWARDES; J. O'REGAN; E. RUTTER; *and others.*

(1) Let R, K, V be middle points of Qq, Pp, pq. Then, since pR, OV, qK are all parallel to axis, we see, by construction, RT = qT; and therefore TQ = 3TR. Also, since pq, RK are evidently parallel, and R, K are middle points of qQ, pP; therefore RK, PQ are parallel, and therefore QP = 3RK = 3pq.



(2) Since PQ, pq are parallel, and V is the middle point of pq, OV produced, bisects PQ in Z, and meets QL in O′, the pole of PQ. Hence we have

$$OO' = LR = 2pR = 2VE = VZ; \text{ therefore } O'V = OZ.$$

(3) $$OO' = 2pR = 2VE = 4OV;$$

therefore $$O'Z = OO' + OV + VZ = 2OO' + OV = 9OV.$$

(4) $$OZ = OV + VZ = OV' + OO' = 5OV.$$

(5) Porduce qO to meet Rp in L; then, clearly, QL is tangent at Q, and is parallel to OR, and similarly tangent at P is parallel to OK. But

QP = 3RK; therefore

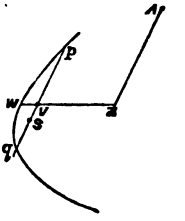
$$\Delta O'PQ = 9\Delta ORK = 9(\Delta ORT + \Delta OKT + \Delta RTK)$$
$$= 9(\Delta RpT + \Delta TqK + \Delta RTK) = 27\Delta pTq = 27\Delta Opq.$$

(6) Let A be intersection of Op, PQ. Then $Ap = qQ = 4qT = 4Op$. Hence, if x', y' be coordinates of A, and x, y those of p, we easily get

$$x' = 4a + 5x, \quad y = \frac{12ax' - 8a^2}{5y'};$$

whence equation to locus is $5y'^2(x' - 4a) = 4a(3x' - 2a)^2$.

If pq coincide with latus rectum, we get $x' = 9a$, $y' = 10a$, which satisfy the above equation. In the second case, OZ is a fixed diameter, and if W be middle point of OV, we readily find $AZ = 5pV$, $WZ = 9WV$. Hence $AZ^2 = \frac{5}{9}m.ZW$, where m is a constant. Hence the locus in this case is a similarly situated parabola.



We can find any number of points on the locus in the first case as follows:—

Draw any focal chord pSq. Draw the diameter WV, and make $VZ = 8VW$. Through Z draw ZA parallel to pSq, and make $AZ = 10VW$, then A is a point on the locus.

6009. (By the late Professor CLIFFORD, F.R.S.)—Find the form of a kite-string under the action of the wind.

Solution by W. J. CURRAN SHARP, M.A.

Let the point of attachment be taken as origin, the axis of x horizontal in the direction of the wind, and the axis of y vertical, and let μ denote the pressure of the wind on a vertical unit of the string, and m its mass; then the equations of equilibrium are

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + \mu\frac{dy}{ds} = 0 \text{ and } \frac{d}{ds}\left(T\frac{dy}{ds}\right) + mg = 0;$$

therefore

$$T\frac{dx}{ds} + \mu y = a \text{ and } T\frac{dy}{ds} + mgs = b,$$

(where $a = P\cos\alpha$, $b = P\sin\alpha$, P being the pull at the point of attachment, and α its inclination to horizon); hence the differential equation to the curve required is

$$\frac{dy}{dx} = \frac{mgs - b}{\mu y - a} \text{ or } (\mu y - a)\frac{d^2y}{dx^2} + \mu\left(\frac{dy}{dx}\right)^2 = mg\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}};$$

hence we have

$$\rho = \frac{(\mu y - a)\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{mg\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} - \mu\left(\frac{dy}{dx}\right)^2}.$$

Thus, at the point whose height is h, and where the inclination of the string to the horizon is θ,

$$\rho = \frac{(\mu h + a)}{mg\cos^2\theta - \mu\sin^2\theta\cos\theta}.$$

So that, for an inflexion, $\cos\theta = 0$ or $mg\cos\theta = \mu\sin^2\theta$.

NOTE ON QUESTION 6009. *By the* EDITOR.

This question was several times referred to in Prof. CLIFFORD's letters to the EDITOR. In a letter bearing date, "Hill's Court, Exeter; S. Giles, 1863," he writes of it as follows:—"I have had in my mind, almost from the time I began to fly kites (and I have not yet left off), the problem of finding the form of a kite-string under the action of the wind. On a rough trial, the other day, the intrinsic equation seemed not very difficult to obtain; if I get any result, I will send it to you hereafter." In the same letter, he adds what, though unconnected with the problem, is not without interest:—"I have been trying to construct a second interpretation of mechanical equations similar to that of tangential coordinates, but have failed hitherto. Being a firm believer in the duality of symbols, I should look upon complete failure as a proof that our symbolical system is wrong. You will be amused by my attempts at obtaining a method of inventing problems by the dozen."

At the EDITOR'S request, Mr. SHARP has supplemented his solution of this interesting problem by the following solution, obtained, as suggested above by Prof. CLIFFORD, by means of the intrinsic equation to the curve.

If the horizontal tangent be taken as the primary line, the equations of equilibrium are $\frac{d}{ds}(T\cos\theta) = \mu\sin\theta$, $\frac{d}{ds}(T\sin\theta) = mg$,

where μ is, as before, the pressure of the wind on a vertical unit of the string. Hence we have

$$T\sin\theta = mgs, \text{ if } s \text{ and } \theta \text{ vanish together,}$$

$$\frac{d}{ds}(mgs\cot\theta) = \mu\sin\theta, \quad \therefore \quad \frac{ds}{d\theta} = s\,.\,\frac{mg\,d\theta}{\sin\theta\,(mg\cos\theta - \mu\sin^2\theta)};$$

and these are inflexions when $\theta = 0$, and when $mg\cos\theta - \mu\sin^2\theta = 0$.

$$\text{Also} \quad \log s = \tfrac{1}{2}\log\frac{1-\cos\theta}{(1+\cos\theta)\,(mg\cos\theta - \mu\sin^2\theta)}$$

$$- \frac{mg}{2\,(m^2g^2+4\mu^2)^{\frac{1}{2}}}\log\frac{2\mu\cos\theta + mg - (m^2g^2+4\mu^2)^{\frac{1}{2}}}{2\mu\cos\theta + mg + (m^2g^2+4\mu^2)^{\frac{1}{2}}} + C,$$

$$\text{or} \quad s^2 = C\frac{1-\cos\theta}{(1+\cos\theta)\,(mg\cos\theta - \mu\sin^2\theta)}\,.\left[\frac{2\mu\cos\theta + mg + (m^2g^2+4\mu^2)^{\frac{1}{2}}}{2\mu\cos\theta + mg - (m^2g^2+4\mu^2)^{\frac{1}{2}}}\right]^{\frac{mg}{(m^2g^2+4\mu^2)^{\frac{1}{2}}}},$$

where C must be determined from the initial conditions.

6206. (By H. MCCOLL, B.A.)—Given (1) that, whenever the statements a, b, x are either all three true or all three false, then the statement c is false and y true, or else c is true and y false; (2) that whenever d, e, y are either all three true or all three false, then the statement a is false and x true, or a is true and x false. When may we infer from these premises that either x or y is true?

I. *Solution by* C. J. Monro, M.A.

In Boole's forms, the question may be stated as follows:—

If $(abx+a'b'x')(cy+c'y')=0$, $(ax+a'x')(dey+d'e'y')=0$ .....(1, 2), when is $x'y'=0$? Clearly, from (1), when $a'b'c'=1$; and, from (2), when $a'd'e'=1$; that is, when a, b, and c, or a, d, and e, or all, are false.

To see whether otherwise, make $x'y'=t'$; then (IX. 9) $Et+Ft'=0$, where F is $a'b'c'+a'd'e'$, the coefficient of $x'y'$ in the sum of (1) and (2), and E is the product of the coefficients of xy, xy', $x'y$. As E contains aa', $Ft'=0$; which means that the truth of $a'b'c'$, or $a'd'e'$, or (VI. 5, IX. 2) both, implies the truth of x or y, or both. So the above answer exhausts the data.

II. *Solution by* Elizabeth Blackwood.

Adopting the Proposer's method and notation, the data are expressed by the complex statement

$$(abx+a'b'x' : c'y+cy')\ (dey+d'e'y' : a'x+ax');$$

hence $xy' : (a+b+c)(d+e+a)$, that is, $x'y' : a+(b+c)(d+e)$;

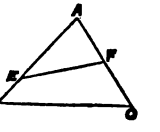
hence, by transposition (*i.e.* contra-position) $a'(b'c'+d'e') : x+y$.

The above solution is exhaustive, and contains every step of the working necessary for obtaining the required antecedent of $x+y$.

6143. (By C. B. S. Cavallin, M.A.)—Find the average area of the triangle cut off by a random line from an equilateral triangle.

Solution by Prof. Matz; Prof. Seitz; *and others.*

Let ABC be the equilateral triangle, EF the random line. Owing to the regularity of the triangle, it is necessary to consider only those positions of the random line in which it cuts AB and AC, and the angle AEF is less than the angle B.



Let $AB=a$, $AE=x$, $\angle AEF=\theta$, and area $AEF=u$.

Then we have $u=\dfrac{x^2\sqrt{3}\sin\theta}{4\sin(\frac{1}{3}\pi+\theta)}$; and the required average is

$$\Delta=\frac{\int_0^{\frac{1}{3}\pi}\int_0^a u\sin\theta\,d\theta\,dx}{\int_0^{\frac{1}{3}\pi}\int_0^a \sin\theta\,d\theta\,dx}=\frac{\sqrt{3}}{2a}\int_0^{\frac{1}{3}\pi}\int_0^a\frac{x^2\sin^2\theta\,d\theta\,dx}{\sin(\frac{1}{3}\pi+\theta)}$$

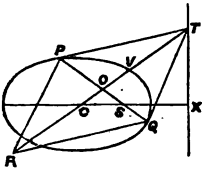
$$=\frac{a^2\sqrt{3}}{6}\int_0^{\frac{1}{3}\pi}\frac{\sin^2\theta\,d\theta}{\sin(\frac{1}{3}\pi+\theta)}=\frac{a^2\sqrt{3}}{24}(3\log 3-2).$$

6115. (By A. J. C. Allen, B.A.)—TP, TQ are tangents at the ends of a focal chord PSQ of an ellipse, and the parallelogram TPRQ is completed; find the locus of R.

Solution by G. Turriff, M.A.; G. Heppel, M.A.; *and others.*

The tangents at P and Q intersect in the directrix; and, putting $\angle RCX = \theta$, we have

$$CT \cdot CO = CV^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta};$$



also $CT - CR = 2CO$;

therefore $CT^2 - CT \cdot CR = 2CT \cdot CO$,

and $$CT = -\frac{a}{e\cos\theta}.$$

Hence, if $CR = \rho$, we get

$$\frac{a^2}{e^2\cos^2\theta} + \frac{a\rho}{e\cos\theta} = \frac{2a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}; \quad (a+ex)(b^2x^2 + a^2y^2) = 2ab^2e^2x^2.$$

6233. (By C. Morgan, B.A.)—Pp, Pq are tangents to a parabola, and the bisectors of the angles pPq, PSp, PSq cut pq, Pp, Pq in f, K, K′ respectively; prove that Pf, KK′ intersect at right angles.

Solution by A. Anderson, B.A.; J. S. Jenkins, L.C.P.; *and others.*

Now $\angle KSK' (= PSp) = TPT'$,

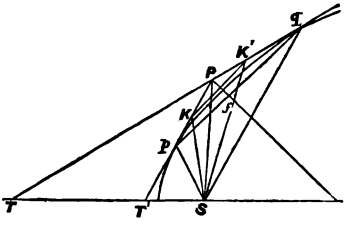
where $\angle KSK' + KPK' = TPT' + KPK' = 2$ rt. angles;

therefore S, K, P, K′ lie on a circle, and $\angle PK'K = PKK'$ (since PSK = PSK′);

therefore PK′ = PK;

and since $\angle KPK'$ is bisected by Pf, and $\angle PK'K = \angle PKK'$,

therefore Pf intersects KK′ at right angles.



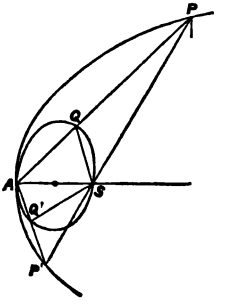
6239. (By T. Morley, L.C.P.)—The minor axis SA of an ellipse is equal to the distance between the foci; PSP′ is a focal chord of a parabola, whose focus is S and vertex A; AP, AP′ cut the ellipse in Q, Q′; prove that QSQ′ is a right angle.

Solution by R. KNOWLES, B.A., L.C.P. ; C. H. SAMPSON ; *and others.*

Let SQ bisect the angle ASP ; then, if
$SQ = \rho$, $SP = r$, $SA = a$, $\angle PSA = 2\phi$,
we have $\rho(a + r)\sin\phi = ar \sin 2\phi$,

or
$$\frac{2\cos\phi}{\rho} = \frac{1}{a} + \frac{1}{r} = \frac{1}{a} + \frac{1 + \cos 2\phi}{2a}$$
$$= \frac{1 + \cos^2\phi}{a},$$

and the locus of Q is $2ax = 2x^2 + y^2$,
which represents an ellipse whose minor axis is SA, equal to the focal distance; therefore
$$\angle QSQ' = \tfrac{1}{2}PSP' = \tfrac{1}{2}\pi.$$



5763. (By Professor SYLVESTER, F.R.S.)—Prove that
$$\Sigma_{i=0}^{i=\infty} \frac{(-)^i x^{\frac{1}{2}(i^2+i)}}{(1-x^k)(1-x^{k+1})\ldots(1-x^{k+i})},$$
when k is any positive integer, is a rational integral function of the degree k^2-k in x. Thus, for example,
$$\frac{1}{1-x} - \frac{x}{(1-x)(1-x^2)} + \frac{x^3}{(1-x)(1-x^2)(1-x^3)} \ldots\ldots = 1,$$
$$\frac{1}{1-x^2} - \frac{x}{(1-x^2)(1-x^3)} + \frac{x^3}{(1-x)(1-x^2)(1-x^3)} \ldots\ldots = 1 - x + x^2.$$

Solution by W. J. CURRAN SHARP, M.A.

Let u_k denote $\Sigma_{i=0}^{i=\infty} \frac{(-)^i x^{\frac{1}{2}(i^2+i)}}{(1-x^k)(1-x^{k+1})\ldots(1-x^{k+i})}$; then

$$u_1 = \frac{1}{1-x} - \frac{x}{(1-x)(1-x^2)} + \frac{x^3}{(1-x)(1-x^2)(1-x^3)}$$
$$= \frac{-y}{1-y} - \frac{y^2}{(1-y)(1-y^2)} - \frac{y^3}{(1-y)(1-y^2)(1-y^3)}, \text{ \&c., if } y = \frac{1}{x}$$
$$= 1 + \frac{1}{1-y} + \frac{1}{(1-y)(1-y^2)} + \text{\&c.}$$
$$- \frac{1}{1-y} - \frac{1}{(1-y)(1-y^2)} - \frac{1}{(1-y)(1-y^2)(1-y^3)} - \text{\&c.} = 1 ;$$

$$\frac{u_{k+1}}{1-x^k} = \frac{1}{(1-x^k)(1-x^{k+1})} + \ldots + \frac{(-)^{i-1} x^{\frac{1}{2}(i^2-i)}}{(1-x^k)(1-x^{k+1})\ldots(1-x^{k+i})} + \text{\&c.},$$

$$x^k u_k = \frac{x^k}{1-x^k} + \ldots\ldots\ldots\ldots + \frac{(-)^i x^{\frac{1}{2}(i^2+i)+k}}{(1-x^k)(1-x^{k+1})\ldots(1-x^{k+i})} + \text{\&c.},$$

therefore $u_{k+1}(1-x^k)^{-1} + x^k u_k = x^k(1-x^k)^{-1} + u_k$,

therefore $u_{k+1} = (1-x^k)^2 u_k + x^k$.

Therefore, since u_1 is of the 0th degree, u_2 will be an integral function of the 2nd degree, u_3 an integral function of 6th degree, and u_k an integral function of the $0+2+4+6+\ldots+2(k-1) = (k^2-k)$th degree in x.

5764. (By Dr. HIRST, F.R.S.)—The projections, from a centre A, upon a fixed axis a, of n given lines $a_1, a_2 \ldots a_n$ in space, form a range which is homographic with a given range $A_1, A_2 \ldots A_n$. Prove that (1) when $n = 4$, the locus of A is a quartic surface having a for a double line, and passing through each of the four given lines, as well as through each of the eight lines which can be drawn to intersect three of the latter and the axis a; (2) when $n = 5$, the locus of A is a curve of the seventh order, which cuts each of the five given lines four times, and the axis a six times; and (3) when $n = 6$, there are four positions of the centre A.

Solution by J. J. WALKER, M.A.

1. Taking a as axis of x', let $a_1 \ldots a_4$ be determined by the planes $l'_1 \equiv x' + B_1 y' + C_1 z' = x' + n'_1 = 0$, and $m'_1 \equiv B'_1 y' + C'_1 z' + 1 = 0 \ldots$, the centre A being (x, y, z). The plane through A and a_1 cuts a in the point $x'_1 = l_1 : m_1$, where $l_1 \equiv x + B_1 y + C_1 z = x + n_1$, $m_1 \equiv B'_1 y + C'_1 z + 1$. The locus of A is, therefore,

$$(l_1 m_2 - l_2 m_1)(l_3 m_4 - l_4 m_3) + k'(l_1 m_3 - l_3 m_1)(l_2 m_4 - l_4 m_2) \ldots\ldots\ldots (1),$$

which is of the form $u_2 x^2 + (u_3 + u'_2) x + u_4 + u'_3 + u''_2 = 0 \ldots\ldots\ldots\ldots (2)$,

the suffixes indicating the orders of homogeneous functions of y, z. The form (2) shows that a is a double line on the quartic surface, locus of A; and the form (1) that the surface passes through each of the four lines $a_1 \ldots a_4$. Either of the lines meeting a and any triad of these four will meet the surface in *five* points, since it meets the surface twice in a, and must therefore lie wholly in the surface.

2. When $n = 5$ the locus of A is the curve of intersection of (1), with

$$(l_1 m_2 - l_2 m_1)(l_3 m_5 - l_5 m_3) + k(l_1 m_3 - l_3 m_1)(l_2 m_5 - l_5 m_2) = 0,$$

or $$v_3 x^2 + (v_3 + v'_2) x + v_4 + v'_3 + v''_2 = 0 \ldots\ldots\ldots\ldots (3).$$

These two quartic surfaces have in common the double line a (counting, therefore, as *four*), the lines a_1, a_2, a_3, and the pair meeting them and a. Their curve of intersection can therefore be met by a plane in only $16-4-3-2$, or 7 points. The line a_4 lying on the surface of (1) is cut by the surface (3) in four points, which must be points in this curve of order 7; and similarly a_5, and by symmetry the other lines of the five. If x is eliminated from (2) and (3), the result is $u_{12} \ldots + u_3 = 0$,—the degree of which is accounted for by the fourfold line a not being included,—a curve passing eight times through the origin, two of which intersections are accounted for by the lines meeting a and each of the triad $a_1 a_2 a_3$. The remaining six passages through the origin indicate so many intersections

of a by the curve, locus of A. Hence, any plane through a determines a single point on that locus. In fact, such a plane cuts the surfaces (2), (3) in two conics, three of the points common to which are those in which the plane is met by $a_1 a_2 a_3$, while the fourth is the single point on the curve, locus of A, determined by the plane.

3. If therefore, lastly, $n = 6$, the positions of the centre A will be any points common to the curves of intersection of (2), (3), and of the former with a third quartic surface

$$(l_1 m_2)(l_3 m_6) + k''(l_1 m_3)(l_2 m_6) = 0, \text{ or } w_2 x^2 + \ldots + w''_2 = 0 \ldots\ldots(4),$$

and, by what has been remarked above, these points may be conveniently determined by planes through a. Substituting λy for z in the equations (2), (3), (4), and discarding the factor y^2, the condition that the three conics—which are known to have three points in common, viz., those in which the plane $\lambda y - z$ meets the triad of lines $a_1 a_2 a_3$—shall have a fourth common point of intersection, gives, on restoring $z : y$ for λ,

$$k(124)(356) + k'(125)(364) + k''(126)(345) + k'k''(134)(256) + k''k(135)(264) + kk'(136)(245) = 0,$$

a homogeneous function of the fourth order in yz.....(5),

where (124) stands for the determinant $\begin{vmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_4 \\ n_1 & n_2 & n_4 \end{vmatrix}$; and the other symbols for similar determinants. Hence there are four positions of the centre A, determined by the four planes through a which form the locus of the above equation.

It may be verified *à posteriori*—what is evident from geometrical considerations—that no one of the equations (1), (3), (4), (5) is altered by changing k into $k+1$..., while interchanging 1 and 3 in the suffixes; or into $1 : k$..., interchanging 2 and 3; or into $k : 1+k$..., interchanging 1 and 2.

5991. (By the Editor.)—A wire hoop is cut at random into three pieces; find the respective probabilities that the three pieces will, when straightened, admit of being formed into (1) a triangle of any kind, (2) an acute-angled triangle.

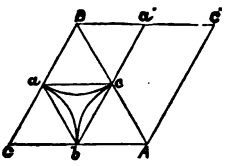
Solution by C. J. Monro, M.A.

The only conditions are that the three quantities are positive and their sum constant. The same is true of the trilinear coordinates of a point inside an equiangular triangle of reference; and, since the probabilities of range in the one case, and the area in the other, vary merely as the product of any two of the three differentials, the values required will be those of certain areas as fractions of the area of the triangle.

Let x, y, z be the coordinates, then the areas will be bounded by the lines given by unlike signs in

$$x \pm y \pm z = 0, \quad x^2 \pm y^2 \pm z^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1, 2);$$

and since the three pieces may be equal, the area will be the central one in each case.



ABC being the triangle, equations (1) give another triangle *abc*, with corners bisecting the sides of ABC, and its area gives the chance ¼.

To interpret (2), complete the parallelogram CAC'B. C'A, C'B are asymptotes to $x^2+y^2=z^2$, as is evident on writing this

$$2(y+z)(z+x)=(x+y+z)^2;$$

for $y+z$, $z+x$ represent C'A, C'B, and $x+y+z$ the line infinity. The curve obviously bisects the sides CA, CB, and with the two others forms a tricuspid figure; the arcs touching, though, of course, with finite curvature. Now *a*B, and *bc* produced so as to meet BC' in *a'*, are ordinates to an asymptote; so we have the area *ba*B*a'*, with one curved side, $=AC'a'b\ (\log C'B-\log C'a')$, $=\Delta\log 2$, if Δ be the area of the triangle of reference. Therefore C*ab* (curved) is $\Delta(1-\log 2)$; and the tricuspid area, as a fraction of Δ, is $3\log 2-2$, the probability obtained by the Editor (*Reprint*, Vol. II., p. 74) and by De Morgan (Vol. V., p. 33) in solutions of Question 1100. In fact, the above solution might be considered a re-interpretation of the Editor's construction; my BC*ba'* being a parallel-projection of his ABDC. See also (Vol. VI., pp. 83-87) Prof. Clifford's paper on the probability of segments which would make a polygon of *n* sides.

The advantage of the present enunciation is that, while every one can see it to be mathematically equivalent to that of Quest. 1100, which supposes the division of a finite straight line, it leaves no doubt that the three segments are symmetrically related in respect of the probabilities of their lengths. But it may be worth while to show directly, what may be thought to have been too implicitly assumed, that this symmetry is true of the finite line. In a line of unit length, $2(1-u)\,du$ is the chance that the length of a specified segment is between u and $u+du$. For, as to the middle one, $1-u$ is the chance that the first cut leaves room for a second at such a distance to the right, and the like to the left, making $2(1-u)$ together; and du is the chance that the second is at such a distance in either case. As to an end segment, $2du$ is the chance that an unspecified cut is made at a distance between u and $u+du$ from a specified end, and $1-u$ that the other is made further off.

6252. (By Dr. Hopkinson, F.R.S.)—It is found that a Leyden jar loses 0·000001 of its charge per second by conduction through the glass. The specific inductive capacity of the glass of which it is made is 8. Show that the resistance of a cubic centimetre of the glass is roughly 14×10^{17} ohms, having given that one electro-magnetic unit of quantity is equal to 3×10^{10} electro-static units, and that one ohm is equal to 10^9 electro-magnetic units.

Solution by Professor G. Carey Foster, F.R.S.

Let R = resistance of jar to conduction between coatings, ρ = specific resistance of glass, S = electrostatic capacity of jar, σ = specific inductive

capacity of glass, and V = difference of potentials between coatings: then we have $\frac{dQ}{dt} = \frac{V}{R} = \frac{Q}{RS} = \frac{4\pi Q}{\rho\sigma}$, since $\rho\sigma = 4\pi RS$;

that is, $\rho = \frac{4\pi Q}{\sigma} \cdot \frac{dt}{dQ}$, or, with the given values, $\rho = \frac{1}{9}\pi \times 10^6$

in electro-static measure.

To reduce to ohms, multiply by $(3 \times 10^{10})^2$, and divide by 10^9; then

$$\rho = \tfrac{1}{9}\pi \times 9 \times 10^6 \times 10^{20} \times 10^{-9} = 14{\cdot}1\ldots \times 10^{17}.$$

6253. (By B. Williamson, F.R.S.)—If two straight lines in a moving plane area always touch the involutes to two circles, prove that any other straight line in the moving area will always touch the involute to a circle.

Solution by Professor Townsend, F.R.S.

This pretty property follows immediately from two others, both of which are well-known,—viz., (1) That when, of a figure of invariable form moving in a plane, two of the lines envelop any curves in the plane, the several lines all envelop corresponding curves in the plane, the corresponding normals to which all pass through a common point in the plane; and (2) That when, of a figure of invariable form moving in a plane, two of the lines envelop circles in the plane, the several lines all envelop circles in the plane, the centres of which all lie on a common circle with the intersection of the corresponding radii in the plane. From these combined, the property in question is an obvious and immediate consequence.

In the first of the above properties, the common point of concurrence of the several normals, in any position of the figure, is obviously the momentary centre of motion, corresponding to the position of the figure in the plane; and in the second, the common circle of position of the several centres, throughout the entire motion, is obviously that passing through those of the two given circles and through the point of concurrence of the parallels through them to the two given lines, in any position of the figure in the plane.

6224. (By Professor Morel.)—Eliminer ϕ et ψ de l'équation

$$Z = x\phi(z) + y\psi(z).$$

Solution by R. Tucker, M.A.; Prof. Scott, M.A.; *and others.*

$\frac{dZ}{dx} = \phi + x\phi' + y\psi'$, $\frac{dZ}{dy} = x\phi' + \psi + y\psi'$, therefore $\frac{dZ}{dx} - \frac{dZ}{dy} = \phi - \psi$;

therefore $\frac{d^2Z}{dx^2} - \frac{d^2Z}{dx\,dy} = \phi' - \psi' = \frac{d^2Z}{dx\,dy} - \frac{d^2Z}{dy^2}$, *i.e.*, $\frac{d^2Z}{dx^2} - 2\frac{d^2Z}{dx\,dy} + \frac{d^2Z}{dy^2} = 0$.

4549, 4566, 4578, 4579. (By C. LEUDESDORF, M.A., and E. B. ELLIOTT, M.A.)—A bag contains mn balls, each equally likely to be of any one of m different colours. All but m are drawn out at random, and found to be $n-1$ of each colour. Find the chance that the remaining m are (1) one of each colour, (2) all of one colour.

I. *Solution by* T. J. SANDERSON, M.A.; E. B. ELLIOTT, M.A.; *and others.*

There is no presumption in favour of any one colour, that is, all colours are equally likely for each of the remaining m balls. Hence the number of all possible changes that may be presented is m^m. Now, the number of ways in which m different colours may be presented, = number of permutations of m things, is $m!$; hence the chance that we find one of each colour is $\frac{m!}{m^m}$. Again, the number of ways in which all can be of one is m; hence the chance that all are of one colour is $\frac{m}{m^m} = m^{1-m}$.

The solutions to Questions 4549 ($m=3$), 4566 ($m=4$), 4579 ($m=5$) follow at once, and give, for the two cases, the following results:—

$$\frac{3!}{3^3} = \frac{2}{9}, \quad \frac{3}{3^3} = \frac{1}{9}; \quad \frac{4!}{4^4} = \frac{3}{32}, \quad \frac{4}{4^4} = \frac{1}{64}; \quad \frac{5!}{5^5} = \frac{24}{625}, \quad \frac{5}{5^5} = \frac{1}{625}.$$

II. *Solution by* C. LEUDESDORF, M.A.

1. Take first the simpler case where $m = 3$ (Quest. 4549). The 10 possible hypotheses may be grouped together under three heads:—

$$(a)\ (n, n, n), \quad (b)\ (n+1, n, n-1), \quad (c)\ (n+2, n-1, n-1);$$

i.e., there may be n balls of each colour; or $n+1$ of one colour, n of second, $n-1$ of the third, &c. Only one hypothesis is of the type (a); six fall under the head (b), and three under (c). The chances of drawing $n-1$ balls of each colour, supposing (a), (b), (c) in turn to be true, are

$$\frac{6\,(n!)^3\,3n-3!}{(n-1!)^3\,3n!}, \quad \frac{3n+1!\,n!\,n-1!\,3n-3!}{(n-1!)^3\,3n!}, \quad \frac{(n-1!)^2\,n+2!\,3n-3!}{(n-1!)^3\,3n!};$$

or $\frac{6n^3}{N}, \quad \frac{3n^2\,(n+1)}{N}, \quad \frac{n\,(n+1)\,(n+2)}{N}, \quad$ where $N = 3n\,(3n-1)\,(3n-2)$.

If we multiply these by the factors 1, 6, 3 respectively, and add, the result is $\frac{3n\,(3n+1)\,(3n+2)}{N}$; so that the required chances are

$$(1)\ \frac{6n^3}{3n\,(3n+1)\,(3n+2)} = \frac{n^3}{C_3^{3n+2}}, \quad \text{and} \quad (2)\ \frac{n\,(n+1)\,(n+2)}{3n\,(3n+1)\,(3n+2)} = \frac{C_3^{n+2}}{C_3^{3n+2}}.$$

2. Proceeding now to the general case, it is seen that the number of possible hypotheses may be very large; but as we are only concerned with the first and the last of these—those, namely, in which the balls are supposed to have been (1) n of each colour, and (2) $n-1$ of all the colours but one, of which colour are the $n+m-1$ remaining balls; and as the sum of the chances, supposing each hypothesis in turn true, must be unity, we may write down the chances required precisely as in the particular case.

They are $\dfrac{(n!)^m\, nm-m!\, m!}{mn!\,(n-1!)^m} + \dfrac{C_m^{mn+m-1}}{mn!\,(n-1!)^m} = \dfrac{n^m}{C_m^{mn+m-1}}$,

and $\dfrac{(n-1!)^{m-1}\, nm-m!\, n+m-1!}{mn!\,(n-1!)^m} + \dfrac{C_m^{mn+m-1}}{mn!\,(n-1!)^m} = \dfrac{C_m^{n+m-1}}{C_n^{mn+m-1}}$.

3. In the above solution, the words of the question, "each equally likely, &c.," have been taken as meaning that the relative proportion of the number of balls of one colour to the number of balls of another colour is unknown, and that any hypothesis as to this is *à priori* equally probable. And in this sense I proposed Questions 4549, 4566, 4579. But the words are ambiguous, as is apparent from various solutions of these questions, which the EDITOR has kindly allowed me to see. One of these starts evidently from the above assumption, and arrives at the above results; but in others, the words are taken as meaning that the chance *à priori* that any given ball is of any given colour is m^{-1}, and the results found are $m! + m^m$ and $m + m^m$ respectively. Taking case (1), it is seen that, with this interpretation of the words, the hypotheses (a), (b), (c) are not *à priori* equally probable; in fact, their *à priori* probabilities are

$$3n! + (n!)^3, \quad 3n! + n+1!\, n!\, n-1!, \quad \text{and} \quad 3n! + (n-1!)^3\, n+2!.$$

If the chances found in (1) be multiplied by these factors, as well as by the factors 1, 6, 3, and added together, we find

$$\frac{3n!}{N\,(n-1!)^3}(1.6+6.3+3.1),$$

and the required chances are $\frac{6}{37}$ and $\frac{3}{37}$.

III. *Solution by* S. TEBAY, B.A.; A. E. DRINKWATER, M.A.; *and others.*

Let $n_1+n_2+\ldots+n_r = m$; then, before the drawing took place, the bag might have contained the following colours: $n-1+n_1$, $n-1+n_2$, &c. to r terms, and $n-1$, $n-1$, &c. to $m-r$ terms; and the number of ways in which the observed event could have happened from this particular hypothesis, is

$$\frac{n-1+n_1!\; n-1+n_2!\ldots n-1+n_r!}{(n-1!)^r\, n_1!\, n_2!\ldots n_r!}.$$

Let $T_{m,r}$ be the sum of all such terms, subject to the condition (1), and let $S_{m,r} = T_{m,r}\dfrac{m!}{r!\,m-r!}$; then the chance that the observed event was the result of this hypothesis, is

$$P_{m,r} = \frac{S_{m,r}}{S_{m,1}+S_{m,2}+\ldots+S_{m,m}},$$

which is the *à priori* probability that the remaining m balls consist of r different colours. These colours may be presented in $\dfrac{m!}{m-r!}$ ways; and the whole number of changes being m^m, the chance that the remaining m balls are of r different colours, is

$$P_{m,r}\cdot\frac{m!}{m-r!}\cdot m^{-m}.$$

Let $m = 3$; then we can take $n_1=3$, $n_1+n_2=3$, $n_1+n_2+n_3=3$, giving

$$S_{3,1} = \frac{n+2\,!}{n-1\,!\;3\,!} \cdot \frac{3\,!}{1\,!\;2\,!} = \tfrac{1}{2}n\,(n+1)\,(n+2),$$

$$S_{3,2} = 2\,\frac{n\,!\;n+1\,!}{(n-1\,!)^2\,1\,!\;2\,!} \cdot \frac{3\,!}{2\,!\;1\,!} = 3n^2\,(n+1),$$

$$S_{3,3} = \frac{n\,!\;n\,!\;n\,!}{(n-1\,!)^3\,1\,!\;1\,!\;1\,!} \cdot \frac{3\,!}{3\,!\;0\,!} = n^3,$$

$$S_{3,1} + \&c. = \tfrac{1}{2}n\,(3n+1)\,(3n+2),$$

$$P_{3,1} = \frac{(n+1)\,(n+2)}{(3n+1)\,(3n+2)}, \qquad P_{3,3} = \frac{2n^3}{(3n+1)\,(3n+2)};$$

and the chances are (1) $\frac{1}{3}P_{3,3}$, (2) $\frac{2}{3}P_{3,1}$.

Let $m = 4$; then we can take $n_1 = 4$, $n_1 + n_2 = 4$, $n_1 + n_2 + n_3 = 4$, $n_1 + n_2 + n_3 + n_4 = 4$, giving

$$S_{4,1} = \frac{n+3\,!}{n-1\,!\;4\,!} \cdot \frac{4\,!}{1\,!\;3\,!} = n\,C_3^{n+3},$$

$$S_{4,2} = \left[2\,\frac{n\,!\;n+2\,!}{(n-1\,!)^2\,1\,!\;3\,!} + \frac{n+1\,!\;n+1\,!}{(n-1\,!)^2\,2\,!\;2\,!}\right] \frac{4\,!}{2\,!\;2\,!} = \tfrac{1}{2}n^2\,(n+1)\,(7n+11),$$

$$S_{4,3} = 3\,\frac{n\,!\;n\,!\;n+1\,!}{(n-1\,!)^3\,1\,!\;1\,!\;2\,!} \cdot \frac{4\,!}{3\,!\;1\,!} = 6n^3\,(n+1), \qquad S_{4,4} = n^4,$$

$$S_{4,1} + \&c. = \tfrac{1}{2}\,(32n^4 + 48n^3 + 22n^2 + 3n),$$

$$P_{4,1} = \tfrac{1}{2}\,\frac{(n+1)\,(n+2)\,(n+3)}{32n^3 + 48n^2 + 22n + 3}, \quad P_{4,4} = \frac{3n^3}{32n^3 + 48n^2 + 22n + 3};$$

and the chances are (1) $\frac{1}{24}P_{4,4}$, (2) $\frac{5}{12}P_{4,1}$.

Let $m = 5$, then we can take $n_1 = 5$, $n_1 + n_2 = 5$, $n_1 + n_2 + n_3 = 5$, $n_1 + n_2 + n_3 + n_4 = 5$, $n_1 + n_2 + n_3 + n_4 + n_5 = 5$, giving

$$S_{5,1} = \frac{n+4\,!}{n-1\,!\;5\,!} \cdot \frac{5\,!}{1\,!\;4\,!} = n\,C_4^{n+4},$$

$$S_{5,2} = \left[2\,\frac{n\,!\;n+3\,!}{(n-1\,!)^2\,1\,!\;4\,!} + 2\,\frac{n+1\,!\;n+2\,!}{(n-1\,!)^2\,2\,!\;3\,!}\right] \frac{5\,!}{2\,!\;3\,!} = \tfrac{5}{6}n^2\,(n+1)(n+2)(3n+5),$$

$$S_{5,3} = \left[3\,\frac{n\,!\;n\,!\;n+2\,!}{(n-1\,!)^3\,1\,!\;1\,!\;3\,!} + 3\,\frac{n\,!\;n+1\,!\;n+1\,!}{(n-1\,!)^3\,1\,!\;2\,!\;2\,!}\right] \frac{5\,!}{3\,!\;2\,!}$$
$$= \tfrac{5}{2}n^3\,(n+1)\,(5n+7),$$

$$S_{5,4} = 4\,\frac{n\,!\;n\,!\;n\,!\;n+1\,!}{(n-1\,!)^4\,1\,!\;1\,!\;1\,!\;2\,!} \cdot \frac{5\,!}{4\,!\;1\,!} = 10n^4\,(n+1),$$

$$S_{5,5} = n^5, \quad S_{5,1} = \tfrac{1}{24}n\,(5n+1)\,(5n+2)\,(5n+3)\,(5n+4) = \tfrac{1}{24}n\,C_4^{5n+4},$$

$$P_{5,1} = \frac{C_4^{n+4}}{C_4^{5n+4}}, \qquad P_{5,5} = \frac{n^4}{C_4^{5n+4}};$$

and the chances are (1) $\frac{1}{625}P_{5,5}$, (2) $\frac{24}{625}P_{5,1}$.

[Mr. Tebay is of opinion that "there is no ambiguity in the statement of the problem," but that "the first solution lacks the *à priori* consideration upon which the conclusion depends, and that the second solution is too direct, and lacks the conclusion."]

6181. (By DONALD MCALISTER, D.Sc.)—Prove that the nth odd power of an odd number greater than unity can be presented as the difference of two square whole numbers in n different ways.

Solution by Prof. SCOTT, M.A.; G. EASTWOOD, M.A.; *and others.*

Take the odd number N; then $\frac{1}{2}(N\pm1)$, $\frac{1}{2}(N^2\pm1)$, ..., are all integers. The nth odd power is

$$\begin{aligned} N^{2n-1} &= N^{2n-2}\left\{\tfrac{1}{2}(N+1)\right\}^2 - N^{2n-2}\left\{\tfrac{1}{2}(N-1)\right\}^2 \\ &= N^{2n-4}\left\{\tfrac{1}{2}(N^2+1)\right\}^2 - N^{2n-4}\left\{\tfrac{1}{2}(N^2-1)\right\}^2 \\ &= \ldots = \left\{\tfrac{1}{2}(N^{2n-1}+1)\right\}^2 - \left\{\tfrac{1}{2}(N^{2n-1}-1)\right\}^2. \end{aligned}$$

[$N^{2n-1} = y^2 - z^2$ is satisfied in whole numbers, provided that $y-z = N^r$, $y+z = N^{2n-r-1}$, if N, and therefore N^r. N^{2n-r-1}, be an odd number; and r may be any whole number from 0 to $n-1$, giving n solutions. If $r > n-1$, we only get z negative, and the values of y^2, z^2 are repeated.]

6097. (By Professor WOLSTENHOLME, M.A.)—Prove that

$$\int_0^\infty \frac{\log x}{(1+x)^4}\,dx = -\tfrac{1}{6}, \quad \int_0^\infty \frac{1-3x}{(1+x)^5}(\log x)^2\,dx = 1,$$

$$\int_0^\infty \frac{1-10x}{(1+x)^6}(\log x)^3\,dx = -3, \quad \int_0^\infty \frac{1-25x+40x^2}{(1+x)^7}(\log x)^4\,dx = 12, \text{ \&c.;}$$

$$U_{n-2} = \int_0^\infty \frac{a_0 - a_1x + a_2x^2 - \ldots}{(1+x)^{n+1}}(\log x)^{n-2}\,dx = (-1)^n\,\frac{(n-2)!}{2}$$

if $a_0 = 1$, $a_1 = 2^{n-1} - (n+1)$, $a_2 = 3^{n-1} - (n+1)\,2^{n-1} + \frac{1}{2}(n+1)\,n$, and so on; the number of such coefficients being $\frac{1}{2}(n-1)$, or $\frac{1}{2}n$, as n is odd or even.

I. *Solution by* H. STABENOW, M.A.; T. R. TERRY, M.A.; *and others.*

The coefficients in any three of these integrals, as U_{2n-4}, U_{2n-3}, U_{2n-2}, say $b_0, b_1, \ldots, b_k, \ldots, b_p, \ldots, b_m, \ldots, b_{n-2}$; $c_0, c_1, \ldots, c_{n-2}$; $d_0, d_1, d_2, \ldots, d_{n-1}$; are connected by the following relations:—

$$c_{p+1} = (p+2)\,b_{p+1} - (p+1)\,b_p + (2n-1)\,b_p \quad \ldots\ldots\ldots\ldots(1),$$

$$d_{p+1} = (p+2)\,c_{p+1} - (p+1)\,c_p + (2n)\,c_p \quad \ldots\ldots\ldots\ldots(2),$$

with $p = 0 = 1 = 2 = \ldots = (n-2)$,

which are verified by extending decomposition and reduction—as of $m \cdot p^k$ into $p^{k+1} + (m-p)\,p^k$ and of $(m)_{k+1}\,p^k + (m)_k\,p^k$ to $(m+1)_{k+1}\,p^k$—to all terms that contain the factor $(p+1)$ or $(p+2)$, and observing that

$d_{n-1} = n^{2n-2} - (2n)_1(n-1)^{2n-2} + (2n)_2(n-2)^{2n-2} - \ldots \mp (2n)_{n-2}\,2^{2n-2} \pm (2n)_{n-1}$

$= \frac{1}{2}\,[x = -n \text{ and } \Delta x = 1]\,\{\Delta^{2n} \cdot x^{2n-2}\} = 0, \quad \pm nc_{n-1} = 0 \ldots\ldots\ldots(3).$

With the help of (1), (2), (3) we deduce, integrating by parts,

$$U_{2n-1} = \frac{1}{2n-3}\int_0^\infty \left\{ \frac{x(b_0 - b_1 x + \ldots \pm b_{n-2}x^{n-2})}{(1+x)^{2n-1}} (\log x)^{2n-3} \right\}$$

$$- \frac{1}{2n-3}\int_0^\infty (\log x)^{2n-3}\, d_x \frac{x(b_0 - b_1 x + \ldots \pm b_{n-2}x^{n-2})}{(1+x)^{2n-1}}$$

$$= - \frac{1}{2n-3}\int_0^\infty \frac{(c_0 - c_1 x + \ldots \pm c_{n-2}x^{n-2})}{(1+x)^{2n}} (\log x)^{2n-3}\, dx$$

$$\pm \frac{nb_{n-2}}{2n-3} \left\{ \int_0^\infty \frac{x^{n-1}}{(1+x)^{2n}} (\log x)^{2n-3}\, dx \right\} = \dot{J};$$

but, substituting $(y^{-1}-1)$ for x, and $(1-z)$ for y, we find

$$\dot{J} = \int_0^1 (1-y)^{n-1} y^{n-1} \left[\log\left(\frac{1}{y} - 1\right) \right]^{2n-3} dy$$

$$+ \left\{ \int_1^{+1} (1-y)^{n-1} y^{n-1} \left[\log\left(\frac{1}{y} - 1\right) \right]^{2n-3} dy \right\} = \dot{J}_1,$$

and $$\dot{J}_1 = - \int_0^1 (1-z)^{n-1} z^{n-1} \left[\log\left(\frac{1}{z} - 1\right) \right]^{2n-3} dz$$

$$= - \int_0^1 (1-y)^{n-1} y^{n-1} \left[\log\left(\frac{1}{y} - 1\right) \right]^{2n-3} dy;$$

therefore $\dot{J}$ vanishes, and hence

$$U_{2n-1} = - \frac{1}{2n-3} U_{2n-3}; \text{ similarly } U_{2n-2} = - \frac{1}{2n-2} U_{2n-3} \ldots (4, 5).$$

We have also $U_1 = -\frac{1}{2}$ .....(6),

as will be found by integrating in

$$U_1 = -\frac{1}{2} \left\{ \frac{\log x}{(1+x)^2} \right\}_0^\infty + \frac{1}{2}\int_0^\infty \frac{dx}{x(1+x)^2},$$

$$\int_0^\infty \frac{dx}{x(1+x)^2} = \int_0^\infty \left\{ \frac{dx}{x} - \frac{dx}{(1+x)^2} - \frac{dx}{(1+x)^2} - \frac{dx}{1+x} \right\}.$$

The relations (4), (5), (6) are obviously sufficient for the proof.

II. *Solution by* Prof. TANNER, M.A.; Prof. NASH, M.A.; *and others.*

Separate the proposed general integral into two portions, the former extending between the limits 0, 1, the latter ranging between 1 and ∞. Transform the second portion by writing $\frac{1}{x}$ for x, and after reduction

recombine it with the first part. Thus we have the given integral,

$$I = \int_0^1 \frac{a_0 + a_1(-x) + a_2(-x)^2 + \ldots - a_2(-x)^{n-3} - a_1(-x)^{n-2} - a_0(-x)^{n-1}}{(1+x)^{n+1}} \times (\log x)^{n-2}\, dx$$

$$= \int_0^1 \{1 - 2^{n-1}x + 3^{n-1}x^2 - \&c.\ \text{ad. inf.}\} (\log x)^{n-2} dx.$$

[This can be verified by comparing coefficients. See also TODHUNTER'S *Hist. of Math. Theory of Probability*, Art. 830.] Observing that

$$\int_0^1 x^r (\log x)^{n-2} dx = (-)^n \frac{(n-2)!}{(r+1)^{n-1}}$$

or

$$\int_0^1 (r+1)^{n-1} x^r (\log x)^{n-2} dx = (-)^n (n-2)!,$$

we have

$$I = (-)^n (n-2)! \{1 - 1 + 1 - 1 + \ldots \text{ad inf.}\} = (-)^n \cdot \tfrac{1}{2} (n-2)!$$

[Prof. TANNER states that the last step is "not unimpeachable." In fact, the PROPOSER devised the integrals in the Question with the express view to test the much-disputed value $\frac{1}{2}$ for the infinite series

$$1 - 1 + 1 - 1 + \ldots;$$

and it will be seen that, so far as this test goes, the result is justified.]

6189. (By Professor JOHNSON, M.A.)—A pack of cards may be shuffled in the following manner :—Take off with your right hand the top card of the pack (held in your left hand face upward), place the next card on top of it, the next below it, the next on top, and so on till the pack is exhausted. Then, if you take the 13 cards of one suit in order, and shuffle thus 10 times in succession, the cards are restored to their original order. On trial with other numbers of cards, it appeared that, in all cases, as soon as the original top card reappeared on the top of the pack, all the cards are in their original places. Prove this, and find the number of shuffles required for n cards. When an odd number of cards is used, the bottom card does not change its position; the above is therefore the result for 12 cards. But when the number of cards is odd, we obtain a new series of results by placing the second card below the first, the third above, the fourth below, and so on. In particular, 2^n cards or $2^n - 1$ cards require only $n+1$ shuffles. With the full pack the ninth shuffle brings all the aces together, all the deuces together, and so on; and three more shuffles (12 in all) restore the original order. A similar arrangement occurs when we use a pack formed of four equal packets. Thus, using 9 cards from each suit, this occurs on the sixth shuffle; three more shuffles in every case restoring the original order. The number of shuffles required are generally very small; a pack of fifty cards, however, requires 50 shuffles.

Solution by Professor LLOYD TANNER, M.A.

We may suppose that there are an even number (n) of cards. Let these be numbered from 1 to n. The effect of a shuffle is to bring 2 before 1, 3 behind 1, 4 before 2, 5 behind 3, and so on. Thus the orders of the cards before and after shuffling are as follows :—

$$1, 2, \ldots, \tfrac{1}{2}n, \tfrac{1}{2}n+1, \ldots n-1, n\,; \quad n, n-2, \ldots, 4, 2, 1, 3, \ldots n-3, n-1.$$

For a given n the effect of any number of shuffles is most easily found by splitting the substitution just written, into its elementary cyclical substitutions, $(1, n, n-1, n-3, \ldots \tfrac{1}{2}n+1)\,(2, n-2, n-5, \ldots \tfrac{1}{2}n), \ldots$.

For instance, when $n = 52$ there are three cycles of the 12th order, and one each of the 6th, 4th, 3rd, 2nd, and 1st orders respectively. The order of the complete substitution is therefore 12, the L.C.M. of the orders of the elementary cycles. To find the effect of r shuffles, we have only to note the rth cards from 1, 2, &c., respectively, each in its proper cycle; these replace 1, 2, &c., after r shuffles.

A somewhat different mode of viewing the question is desirable when n is not known. Let r_m represent the place of a card *before*, r_{m+1} its place *after* a shuffle. Then we find that, according as r_m is odd or even,

$$2r_{m+1} = n + r_m + 1, \text{ or } n - r_m + 2\,;$$

that is,
$$2\,(2r_{m+1} - 1) = 2n + 1 \pm (2r_m - 1) \ldots\ldots\ldots\ldots (1).$$

In this formula there is no ambiguity, for, r_m being given, only one sign will make r_{m+1} integral. The same remark applies to the following equation, which is derived from (1) :—

$$2^m\,(2r_m - 1) = (2n+1)\,(2^{m-1} \pm 2^{m-2} \pm \ldots \pm 2 \pm 1) \pm (2r_0 - 1) \ldots\ldots (2).$$

This equation gives the position (r_m) of a card after m shuffles, the original position being r_0.

Suppose that, after m shuffles, the card in question comes to its original position, so that $r_m = r_0$. Then (2) becomes

$$(2^m \pm 1)\,(2r_0 - 1) = (2n+1)\,(2^{m-1} \pm 2^{m-2} \pm \ldots \pm 2 \pm 1) = (2n+1)\,.\,\mathrm{M} \ldots (3),$$

where M is an odd number between 0 and 2^m. The smallest value of m which satisfies this equation for all values of r_0 gives the smallest number of shuffles required to reduce the cards to the original order.

Let $r_0 = 1$; then (3) becomes $2^m \pm 1 = (2n+1)\,.\,\mathrm{M}' \ldots\ldots\ldots\ldots (4)$.

If $\phi\,(2n+1)$ be the number of numbers less than $2n+1$, and prime to it, one solution of this equation is $m = \phi\,(2n+1)$; but m may be the result of dividing $\phi\,(2n+1)$ by a power of 2. A nearer limit may be obtained when $2n+1$ is a product of factors each of which is a prime or a power of a prime; viz., taking the ϕ of each factor, the L.C.M. of all these ϕ's is either m or the product of m into some power of 2. The proper value of m may be determined by trial, or by a method given in the *Messenger of Mathematics* (Vol. VIII., p. 13).

Supposing m determined for $r_0 = 1$, it can be shown that (3) is satisfied by the same value of m, whatever r_0 may be. For, by means of (4), (3) becomes
$$(2r_0 - 1)\,\mathrm{M}' = \mathrm{M},$$
which gives a value of M within the prescribed limits, as appears on eliminating M' with the help of (4), and noting that $r_0 \leqq n$. We already know that only one value of M exists. This verifies the statement that, when the top card comes to its original position, the original order is restored.

Examples:—$n = 52$, $2n+1 = 105 = 3.5.7$;

the value of ϕ (105) is 48, while the L.C.M. of ϕ (3), ϕ (5), ϕ (7), is 12. Hence $m = 3$, 6, or 12; and on trial we find that $2^{12}-1$ is, but $2^6 \pm 1$ are not, divisible by 105. Hence $m = 12$.

$n = 2^p$ or $2^p - 1$; (4) becomes $(2^m \pm 1) = (2^{p+1} \pm 1)\,M'$; so that $M' = 1$, $m = p+1$.

$n = 12$, $\quad \phi(2n+1) = \phi(25) = 20$, $\quad m = 5$, 10, or 20.

Since $2^{10}+1$, $= 1025$, is a multiple of 25, and $2^5 \pm 1$ are not, $m = 10$.

Of course, for particular values of r_0, (3) may be satisfied by smaller values of m; indeed, this must be the case unless m is a factor of n. For instance, take $n = 52$, $r_0 = 18$, then (3) is satisfied by $m = 1$, becoming $3.35 = 105$. The same equation is satisfied by $m = 2$, $r_0 = 11$ or 32.

To examine the effect of p shuffles upon a pack containing $2^{p-1}.n$ cards (in particular, the effect of 3 shuffles on a pack of $4n$ cards). Making the proper substitutions, and putting $r_0 = 2^{p-1}r + s$, (2) becomes

$$2^p(2r_p - 1) = (2^p n + 1)\,M \pm (2^p r + 2s - 1), \quad \text{where M is odd and } < 2^p;$$

therefore
$$2r_p - 1 = n.M \pm r + \frac{M \pm (2s-1)}{2^p} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The last term can only be 1 or 0 (since it is integral). When $r \equiv n$ (mod. 2) the last term must be 1 (for the left side is odd). Thus $M = 2^p - 2s + 1$, and the upper sign is to be used. In this case

$$r_p = 2^{p-1}n + 1 + \tfrac{1}{2}(n+r) - s.n.$$

When $r \equiv n+1$ (mod. 2), the last term of (5) must vanish; that is, we must have $M = 2s - 1$, and use the lower sign. Then

$$r_p = s.n - \tfrac{1}{2}(n+r-1).$$

If the pack be arranged in n packets, containing 2^{p-1} aces, deuces, &c. respectively, to each packet belongs one value of r, and to each card in this packet one value of s. The values of r_p just obtained show that after p shuffles the cards of each packet are distributed, each being the nth from its former neighbour. It may also be inferred that, if the final arrangement is to be in sequence, suits attended to, the packets must be arranged thus: at the bottom the aces, then the n's (or "Kings" say), then the deuces, then the "Queens," and so on. Also two consecutive packets must have their suits in opposite orders.

The number of shuffles required to produce this arrangement from the pack arranged in sequences, is $m-p$, m having the signification given in (3).

6169. (By Elizabeth Blackwood.)—If P, Q, R are random points within a sphere of which O is the centre; find the average volume of the tetrahedron OPQR.

Solution by C. J. Monro, M.A.

Naught, by rights; but signless volume is obviously meant. Thanks, then, to the symmetry of the conditions, we may attend only to the cases in which the volume is positive, and, of course, fix one edge through O in direction, and one face through that edge in aspect. Then the

ranges of P, Q, and R are a semidiameter, a semicircle adjacent, and a hemisphere adjacent to this; and their mean positions are at mutually perpendicular distances $\frac{1}{2}$, $\frac{4}{3\pi}$, $\frac{3}{8}$ of the radius. Since the volume is $\frac{1}{3}$ of the product of the distances of P from O, of Q from OP, and of R from OPQ, the mean positions give a mean tetrahedron, $\frac{1}{24\pi}$ in volume.

6231. (By R. Tucker, M.A.)—Eliminate θ and ϕ from the equations

$$\frac{x}{a}\cos\theta + \frac{y}{b}\cos\phi = \frac{x}{a}\sin\theta + \frac{y}{b}\sin\phi = 1 \quad \ldots\ldots(1),$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \mu^2(\sin 2\theta - \sin 2\phi) \quad \ldots\ldots(2).$$

Solution by C. H. Sampson, M.A.; A. L. Selby, B.A.; *and others.*

From (1), $$\frac{x^2}{a^2} + \frac{2xy}{ab}\cos(\theta-\phi) + \frac{y^2}{b^2} = 2;$$

hence, solving for $\frac{x}{a}$ and $\frac{y}{b}$, we have

$$\frac{x}{a} = \frac{\sin\phi - \cos\phi}{-\sin(\theta-\phi)}, \quad \frac{y}{b} = \frac{\sin\theta - \cos\theta}{\sin(\theta-\phi)};$$

therefore $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\sin 2\theta - \sin 2\phi}{\sin^2(\theta-\phi)} = \mu^2(\sin 2\theta - \sin 2\phi);$$

therefore $$\sin^2(\theta-\phi) = \frac{1}{\mu^2} \text{ and } \cos(\theta-\phi) = \frac{\pm(\mu^2-1)^{\frac{1}{2}}}{\mu};$$

therefore $$\frac{x^2}{a^2} \pm \frac{2xy}{ab}\frac{(\mu^2-1)^{\frac{1}{2}}}{\mu} + \frac{y^2}{b^2} = 2.$$

3915. (By J. J. Walker, M.A.)—From points on a diameter of a conic, tangents are drawn and perpendiculars let fall on the chords of contact; prove that the locus of the intersection of these last named lines is a hyperbola having as asymptotes the diameter conjugate to the given one and a perpendicular diameter.

Solution by Professor Wolstenholme, M.A.

The equation of the diameter being $\frac{x}{a\cos\alpha} = \frac{y}{b\sin\alpha} (= m)$, the equations of a chord of contact and of the perpendicular thereon are

$$m\left(\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b}\right) = 1, \quad \frac{x - ma\cos\alpha}{\frac{\cos\alpha}{a}} = \frac{y - mb\sin\alpha}{\frac{\sin\alpha}{b}};$$

therefore $$\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}-m\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha\right)}{\frac{x\cos\alpha}{a^2}+\frac{y\sin\alpha}{b^2}}=\frac{bx\sin\alpha-ay\cos\alpha}{\left(\frac{b}{a}-\frac{a}{b}\right)\sin\alpha\cos\alpha},$$

or $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{b}{a}-\frac{a}{b}\right)=\left(\frac{x\cos\alpha}{a^2}+\frac{y\sin\alpha}{b^2}\right)\left(\frac{bx}{\cos\alpha}-\frac{ay}{\sin\alpha}\right),$$

or, for the locus, $x^2-y^2+\frac{xy}{ab}(a^2\tan\alpha-b^2\cot\alpha)=a^2-b^2$, which denotes a rectangular hyperbola passing through the four foci of the ellipse, and being in fact the locus of the intersection of tangents equally inclined to a given line, which is in this case the direction of the chord of contact (or the perpendicular); whence we get the following well-known proposition:—If from a point O a perpendicular OP be drawn on its polar with respect to a conic whose foci are S, H, then OP will bisect the angle SPH; or, stated in other words, if a tangent P to an ellipse have the pole O with respect to a confocal, OP will be normal at P.

[Mr. Walker's proof is as follows:—If C be the centre and A the vertex of the given diameter, and if OP meet the conjugate diameter in Q, then $PO.OQ=CA^2\sin OCP\cos OCP=$ constant. In the case of the parabola, the locus is easily seen to be a straight line.]

Note on Question 3915. *By the* Editor.

This Question, it has been pointed out, is a particular case of the following theorem:—From different points of a fixed straight line perpendiculars are drawn upon their polars with respect to a given conic; the locus of the feet of these perpendiculars is the circular cubic which is the locus of the points of contact of tangents drawn from a fixed point (the pole of the fixed straight line) to the conics confocal with the given conic. The proposition stated at the end of the foregoing solution proves this at once; but, without assuming that result, let the equations of the fixed line and fixed conic be $$\frac{px}{a^2}+\frac{qy}{b^2}=1,\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and let (X, Y) be any point on the fixed line; then the foot of the perpendicular is given by the equations

$$\frac{x-X}{\frac{X}{a^2}}=\frac{y-Y}{\frac{Y}{b^2}}\ (\equiv\lambda);\quad \frac{xX}{a^2}+\frac{yY}{b^2}=1;\quad \text{also } \frac{pX}{a^2}+\frac{qY}{b^2}=1;$$

and, substituting for (X, Y), we get the two equations

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,\quad \frac{px}{b^2+\lambda}+\frac{qy}{b^2+\lambda}=1;$$

proving that the foot of the perpendicular is a point of contact of a tangent from (p, q) to a conic confocal with the given conic. Eliminating λ, we get the equation of the locus

$$(py-qx)(x^2+y^2-px-qy)=(a^2-b^2)(x-p)(y-q).$$

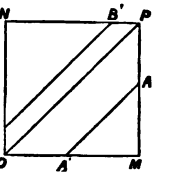
This is, in general, a circular cubic with a crunode at (p, q). When p and q are infinite, it reduces to the rectangular hyperbola

$$(py-qx)(px+qy)+(a^2-b^2)pq=0.$$

5138. (By the Editor).—Two persons make an appointment to meet on a certain day at the Medical Council Office, but without fixing the time further than that it is to be between 2 and 3 o'clock; and each agrees to wait ten minutes for the other. Now, supposing all times between the specified limits for coming to the Office to be equally probable, show, by a general solution, that the odds are 25 to 11 against the meeting taking place.

I. *Solution by* C. J. Monro, M.A.

To state the question generally, take as unit of time the interval allowed for calling. The chance that A calls between the times x and $x+dx$ is dx, and the chance that B calls between the times y and $y+dy$ is dy. Therefore the chance that they meet is $\iint dx\,dy$ between proper limits. If x and y be counted from the beginning of the interval, we have the limits 0 and 1 for each; and if A. and B. wait for the periods a and b, we have the further limits $x-y=-a$ and b; and here inner limits will supersede outer ones. Proceeding geometrically, for variety's sake, draw a unit square OMPN, and chords AA′, BB′, parallel to OP, such that PA = a, OB = b. Then the hexagon OA′APB′B will be measured by the above integral, and will therefore, without danger of introducing geometrical conditions foreign to the data, serve to measure the chance required. This, then, is



$$1-\tfrac{1}{2}(1-a)^2-\tfrac{1}{2}(1-b)^2, \text{ or } a+b-\tfrac{1}{2}(a^2+b^2),$$

and becomes $2a-a^2$ if $b=a$, or $\frac{11}{18}$ if also $a=\frac{1}{6}$; thus the odds against the meeting are 25 to 11.

II. *Solution by* Prof. Seitz, M.A.; Prof. Nash, M.A.; *and others.*

Let x and y be the times after 2 o'clock at which the two persons arrive at the office, and $a=\frac{1}{6}$ of an hour the time each agrees to wait for the other. Then, in order that they may meet, the difference between x and y must not exceed a; hence, the chance that they will meet is

$$\int_0^a\int_0^{x+a} dx\,dy+\int_a^{1-a}\int_{x-a}^{x+a} dx\,dy+\int_{1-a}^{1}\int_{x-a}^{1} dx\,dy$$

$$=\int_0^a (x+a)\,dx+\int_a^{1-a} 2a\,dx+\int_{1-a}^{1}(1+a-x)\,dx=2a-a^2=\tfrac{11}{36}.$$

III. *Solution by the* Rev. H. G. Day, M.A.; J. A. Kealy, M.A.; *and others.*

Let the interval for calling be c hours, and let A resolve to wait m hours; then the chance that B comes before A leaves is $\frac{m}{c}-\frac{1}{2}\frac{m^2}{c^2}$.

Similarly, let B resolve to wait n hours; then the chance that A comes before B leaves is $\frac{n}{c}-\frac{1}{2}\frac{n^2}{c^2}$.

Therefore chance of their meeting is $\frac{m+n}{c}+\frac{1}{2}\frac{m^2+n^2}{c^2}$.

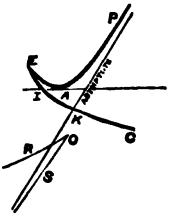
In the given case, we have $m=n=\frac{c}{6}$; therefore the chance is $\frac{11}{36}$, and the odds are accordingly 25 to 11 against the meeting taking place.

1628. (By Professor SYLVESTER, F.R.S.)—Extending the meaning of the word Bicorn to signify the general curve defined by the equations

$$x = \frac{c^3}{\phi^4 + \phi^3}, \quad y = \frac{c^2(2\phi + 3)}{\phi^3 + \phi^2},$$

show that the limiting form of the curve, as c converges towards zero, is a semicubical parabola and its axis extended indefinitely in both directions, forming together a sort of Trident. Explain the mode in which the geometrical passage of the double-horned to the tridentine form takes place, and specify in the Trident the morphological equivalents of the respective four branches of the general curve.

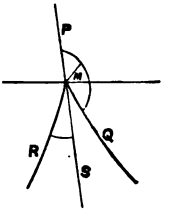
Solution by the PROPOSER.



1. *General Curve.*—The general equation in x, y, c, when c vanishes, becomes

$$x(x^3 - ky^3) = 0,$$

k being a known number (in fact $k = \frac{1}{27}$).



2. *Trident.*—Here E, I, K, A, O come together at M, and the Table of Morphological Equivalents is as follows:—

$M \equiv$ E, I, A, K, O,	$MS \equiv$ OS,
$PMS \equiv$ Asymptote,	$MQ \equiv$ EIKQ,
$MP \equiv$ EAP,	$MR \equiv$ OR.

The triangular space EIA vanishes; O moves up to A; EAP becomes MP; OS becomes MS; EIQ becomes MQ; OR becomes MR; and MQ, MR are the two branches of a semicubical parabola, the asymptote becoming PMS.

6229. (By the EDITOR.)—Supposing the value of a rough diamond to vary as the square of its size; find how such a diamond would be reduced in value if it were broken at random into (1) two pieces, (2) three pieces.

I. *Solution by* J. HAMMOND, M.A.; J. A. KEALY, M.A.; *and others.*

Denoting the parts by the letters x, y, z, the conditions are

$$x + y = 1, \quad x + y + z = 1 \ldots\ldots\ldots\ldots\ldots\ldots (1, 2).$$

The value of the broken diamond is, in the first case, $x^2 + y^2$, and in the second $x^2 + y^2 + z^2$; its mean value is therefore the mean value of the square of the radius vector from the origin to all points of (1) or (2), and the mean reduction in value is $2xy$, or $2(yz + zx + xy)$, with the same limits of integration.

In the first case, $2xy = 2x(1-x)$, and the reduction of value is

$$2\int_0^1 x(1-x)\,dx = \frac{1}{3}.$$

In the second case, we have for the reduction $2\int(yz+zx+xy)\frac{dA}{A}$, where A is the area of $x+y+z=1$ intercepted by the planes of x, y, z; or, since $\frac{dA}{A}$ is the same for the area and its projection,

$$2\iint\{xy+(x+y)(1-x-y)\}\frac{dx\,dy}{\frac{1}{2}}$$

(the integration extending over the triangle $x=0,\ y=0,\ x+y=1$)

$$= 4\int_0^1\int_0^{1-x}[x(1-x)+y(1-x)-y^2]\,dx\,dy$$

$$= 4\int_0^1\left[x(1-x)^2+\frac{(1-x)^3}{6}\right]dx = \frac{1}{3}+\frac{1}{6} = \frac{1}{2}.$$

II. *Solution by* C. J. Monro, M.A.

Let a be the size of the whole, and let kx^2 be the value of size x. The probability that, when the size of two pieces together is constant, the size of either is between x and $x+dx$, is doubtless understood to vary as dx.

1. The expectation is obviously $k\int_0^a\{x^2+(a-x)^2\}\frac{dx}{a} = \frac{2}{3}ka^2$, giving a loss of $\frac{1}{3}$. This is equivalent to an actual division into about $\frac{2}{3}$ and $\frac{1}{3}$.

2. This will be greatly simplified by proceeding geometrically. Let $x+y+z=a$ be the equation to a plane: the chance of a particular range of sizes will be measured by the area of the corresponding range of points as a fraction of the area of the triangle enclosing points with positive coordinates. Let $d\mathrm{S}$ be an elementary area, and S_1 the whole. Then the expectation is $k\int(x^2+y^2+z^2)\frac{d\mathrm{S}}{\mathrm{S}_1}$; or, if r (the coordinates being rectangular, is the distance from the perpendicular (of length $\frac{1}{2}a\sqrt{3}$) drawn to the plane from the origin, $k\int\left(\frac{a^2}{3}+r^2\right)\frac{d\mathrm{S}}{\mathrm{S}_1}$. The second term of the integral will be the square of the radius of gyration of an equilateral triangle of side $\sqrt{2}\,.\,a$, about a normal through its centre, which is $\frac{1}{6}a^2$. So the expectation is $\frac{1}{2}ka^2$, giving a loss of $\frac{1}{2}$. This is equivalent to an actual division into about 13, 5, 2 twentieths.

6013. (By Professor Minchin, M.A.)—If in any point of a body subject to stress the principal stresses consist of two tensions of intensities A and B (A > B) and a pressure of intensity C, show that the maximum intensity of shearing stress is $(AC)^{\frac{1}{2}}$, and find the plane on which it is exerted. If

the principal stresses are a tension of intensity A and two pressures of intensities B and C (C > B), show that the maximum intensity of shearing stress is $(AC)^{\frac{1}{2}}$, and find the plane on which it is exerted.

Solution by J. J. WALKER, M.A.

As preliminary to the solution, consider the semi-diameter r to any point (xy) on a hyperbola $a^{-2}x^2 - b^{-2}y^2 = 1$, and p the perpendicular from centre on the tangent at (xy); and let

$$u^2 = r^{-2}(p^{-2} - r^{-2}).$$

Then $a^4b^4u^2 = (b^4x^2 + a^4y^2)(x^2 + y^2)^1 - (b^2x^2 - a^2y^2)^2(x^2 + y^2)^{-2}$,

and $a^{-2}b^{-2} - u^2 = (a^2x^2 + b^2y^2)(b^2x^2 - a^2y^2)\,a^{-2}b^{-2}(x^2 + y^2)^{-2}$,

an essentially positive quantity, if finite, and which vanishes only when (xy) is the point of contact of an asymptote. The maximum value of u is, therefore, at that limit, when $u^2 = a^{-2}b^{-2}$.

Now, consider the stress-quadric $Ax^2 \pm By^2 - Cz^2 = 1$. The planes across which the force is a simple shear, are those which have the sides of the asymptotic cone $Ax^2 \pm By^2 - Cz^2 = 0$ as normals, and the force across any plane being, in general, proportional to $r^{-1}p^{-1}$ (r being the semi-diameter of the stress-quadric normal to that plane, and p the perpendicular from centre on the tangent plane at the end of r), and in the direction of p, its component parallel to the plane will be $r^{-1}p^{-1}\sin(rp)$, or u, as above. For any one of the simple shears specified, $u^2 = a_1^{-1}b_1^{-1}$, a_1, b_1 being the semi-axes of the hyperbolic section of the stress-quadric made by the plane rp, where r is a side of the asymptotic cone and p the perpendicular at the centre to the tangent plane along that side. Now, lmn being the direction-cosines of the plane of the section,

$$a_1^{-2}b_1^{-2} = (a^2l^2 \pm b^2m^2 - c^2n^2) + (\pm a^2b^2c^2), \text{ where } a^{-2} = A\ldots;$$

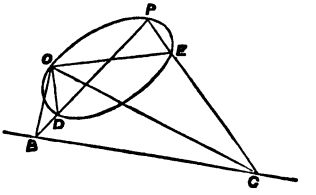
and therefore $a^2b^2c^2(a^{-2}c^{-2} - a_1^{-2}b_1^{-2}) = (b^2 - a^2)l^2 + (b^2 + c^2)m^2$,

an essentially positive quantity when $A > B$, or $(a^2 + b^2)l^2 + (b^2 - c^2)n^2$, which is positive for all values of l, n, if $C > B$. The planes of maximum intensity $(AC)^{\frac{1}{2}}$ of shearing force are therefore those normal to the asymptotic lines $Ax^2 - Cz^2 = 0$.

NOTE ON QUESTION 1409. *By* C. TAYLOR, M.A.

The following theorem is due to the late Professor CLIFFORD:—

Corresponding to every point O on a conic there exists a certain straight line BC without it, such that if chords PD, PE be drawn from any point P on the conic, and be produced to meet BC in B and C, then the angles BOC and DOE are either equal or supplementary. (*Reprint*, Vol. I., pp. 33, 40.)



I propose to show that this affords an illustration of the method of transformation which I have described under the name *Reversion*, in the *Quarterly Journal of Mathematics*, Vol. XIV., pp. 25—39. It is there shown that all angles subtended at the origin O reverse into *equal angles* subtended at the origin ω (p. 33); and hence, that a variable chord of a conic which subtends a right angle at a fixed point O on the curve passes through a fixed point on the normal at O (p. 35), for, if we reverse the conic from O as origin into a *circle*, the variable chord is reversed into a *diameter* of the circle, since it subtends a right angle at the point ω on the circumference.

Hence, conversely, we see that any conic may be reversed into a circle from any point O on the conic as origin *by taking as base-line the polar of the point of concourse of all chords which subtend right angles at* O.

If BC be the line thus related to O in the diagram, the conic reverses into a circle through the reverse origin ω, and (by the nature of reversion) the line at infinity corresponds to BC. Hence, if the points $pde \infty \infty'$ correspond severally to PDEBC; then, since *angles in the same segment of a circle* are equal to one another, therefore

$$\angle d\omega e = dpe = \infty p \infty' = \infty \omega \infty';$$

and, therefore, in the original or obverse figure, the angles DOE and BOC are equal or supplementary, as was to be proved.

Prof. CAYLEY has pointed out (*Reprint*, Vol. I., p. 40) that the theorem in question may be regarded as a limiting case of NEWTON'S *Descriptio Organica*. For if an angle BOD of given magnitude turn about a fixed point O, and if a straight line PDB, regarded as an angle of *zero* magnitude, turn about a fixed point P; then, if the point of concourse B be made to describe a straight line to BC, the point of concourse D will describe a conic through O and P.

If COE be any second position of the constant angle, then, in the diagram,

$$\angle \mathrm{DOE} = \mathrm{DOC} + \mathrm{COE} = \mathrm{DOC} + \mathrm{DOB} = \mathrm{BOC},$$

which is Prof. CLIFFORD'S theorem; for it may be shown that the line BC has a relation to O which is independent of the position of P on the conic.

To prove the theorem in this second form, by Reversion, let an angle of given magnitude α turn about the origin ω, and a straight line about the origin O; and let this line be always *parallel* to one arm of the constant angle, and meet the other in d, the locus of which is evidently a *circle* through O and ω.

Then, in the obverse figure, we have an angle equal to α turning about O, and a straight line turning about ω (or P), and meeting one arm of the angle on a conic through O and ω, and the other arm on BC the base-line, which is the polar of the point of concourse of all chords of the conic which subtend right angles at O.

The theorem of Quest. 1263 (*Reprint*, Vol. XXXII. p. 29) is deduced from the general case of Quest. 1409 above stated, by making D and E the points at infinity on a rectangular hyperbola.

2668. (By N'IMPORTE.) — A writes to B, requiring an answer within n days. It is known that B will be at the address on some one of these days, any one equally likely. It is a p-days' post between A and B. If one in every q letters is lost in transit, find the chance that A receives an answer in time. $(n > 2p)$.

Solution by J. E. A. STEGGALL, B.A.; C. J. MONRO, M.A.; *and others.*

Call *available* days those on which B would write in time if the post did not fail. It is not quite clear whether the pth is to be an available day or not. Say there are $n-2p+\epsilon$ available days, ϵ being 1 or 0. Then the chance required is the product of the chances that A's letter is not lost $\left(1-\frac{1}{q}\right)$, that B calls on an available day $\left(\frac{n-2p+\epsilon}{n}\right)$, and that B's letter is not lost $\left(1-\frac{1}{q}\right)$; in short $\left(1-\frac{1}{q}\right)^2 \frac{n-2p+\epsilon}{n}$. Is it, however, certain B would answer? If w_x is the chance that he is in a good humour on the xth day, the value is

$$\left(1-\frac{1}{q}\right)^2 \frac{1}{n} \Sigma_{p+1-\epsilon}^{n-p} w_x.$$

6267. (By C. LEUDESDORF, M.A.)—In a plane triangle R, r, r_1, r_2, r_3 are the radii of the circumscribed, inscribed, and three escribed circles; prove that, if $(r_2+r_3-r_1)(r_3+r_1-r_2)(r_1+r_2-r_3) = -8r_1r_2r_3$, then $r+4R$ will be equal to the perimeter of the triangle.

Solution by the D. EASTWOOD, M.A.; J. OPENSHAW, M.A.; *and others.*

A simple proof may be immediately obtained by aid of the following relations, first proved in the late T. S. DAVIES's *Horæ Geometricæ*, in the *Ladies' Diary* for 1836:—

$$r_2r_3+r_3r_1+r_1r_2 = s^2, \quad r_1+r_2+r_3 = 4R+r = 2\sigma, \text{ suppose;}$$

for the given condition thus becomes, at once,

$(\sigma-r_1)(\sigma-r_2)(\sigma-r_3) = -r_1r_2r_3$, or $\sigma^3-2\sigma.\sigma^2+s^2.\sigma = 0$, whence $\sigma = s$.

6163. (By Professor MATZ, M.A.)—Prove that the mean distance of all the points of the cardioid-curve from the coordinate origin is $\frac{4}{3}a$.

Solution by G. J. GRIFFITHS, M.A.; Professor NASH, M.A.; *and others.*

The mean distance is $\int_0^s r\,ds \div s$; but, in the cardioid, $r = a(1-\cos\theta)$, and $ds = 2a\sin\frac{1}{2}\theta\,d\theta$; therefore the mean value of r is

$$a\int_0^\pi (1-\cos\theta)\sin\tfrac{1}{2}\theta\,d\theta \div \int_0^\pi \sin\tfrac{1}{2}\theta\,d\theta$$

$$= 2a\int_0^{\frac{1}{2}\pi} \sin^3\theta\,d\theta \div \int_0^{\frac{1}{2}\pi} \sin\theta\,d\theta = 2a.\tfrac{2}{3} = \tfrac{4}{3}a.$$

6187. (By A. Martin, M.A.)—Four cards are taken from a full pack; n other cards being drawn, none of them are aces; find the chance that the first four cards drawn were all aces.

Solution by Prof. Matz, M.A.; W. H. Lowry, M.A.; *and others.*

The final state of the cards is as follows:—There are n exposed cards, none of them aces; and $52-n$ unexposed cards, divided into two lots containing 4 and $52-n-4$ cards respectively. It is required to find the chance that the four aces are all together in the small lot; this chance is clearly $4! \div \{(52-n)(51-n)(50-n)(49-n)\}$, the reciprocal of the number of ways in which 4 cards can be drawn from a pack of $52-n$ cards.

5329. (By Professor Townsend, F.R.S.)—A rigid body, in unconstrained equilibrium in free space, being supposed set in motion by a single impulsive force applied at a definite point of its mass; if the initial motion be a pure rotation, show that—

(a) The several possible lines of impulsive action through the point generate a quadric cone, containing the three perpendiculars from the point on the three central principal planes, and the connector of the point with the centre of inertia of the body.

(b) The several corresponding axes of initial rotation in the body envelope a quadric cone, touching the three central principal planes, and the diametral plane conjugate to the direction of the point with respect to the central ellipsoid of the body.

Solution by the Proposer.

Denoting by M the mass of the body, by abc its three principal arms of central inertia, by xyz the coordinates of the point with respect to its three central principal planes, by F the impulsive force, by $\alpha\beta\gamma$ the direction angles of its action at xyz, by uvw and pqr the components, linear and angular, of the initial velocities produced by its action at and round the centre of inertia of the body, and by $\xi\eta\zeta$ the coordinates of any point on the initial axis of twist resulting from the composition of both motions; then, since, on the hypothesis of initial pure rotation,

$$u+q\zeta-r\eta=0,\quad v+r\xi-p\zeta=0,\quad w+p\eta-q\xi=0;$$

therefore, substituting for uvw and pqr their known dynamical values, we have the three equations

$$\cos\alpha+b^{-2}(z\cos\alpha-x\cos\gamma)\zeta-c^{-2}(x\cos\beta-y\cos\alpha)\eta=0,$$
$$\cos\beta+c^{-2}(x\cos\beta-y\cos\alpha)\xi-a^{-2}(y\cos\gamma-z\cos\beta)\zeta=0,$$
$$\cos\gamma+a^{-2}(y\cos\gamma-z\cos\beta)\eta-b^{-2}(z\cos\alpha-x\cos\gamma)\xi=0;$$

and from these, by elimination, first of $\xi\eta\zeta$, and then of pqr, the relations

$$p\cos\alpha+q\cos\beta+r\cos\gamma=0,\quad\text{and}\quad\xi\cos\alpha+\eta\cos\beta+\zeta\cos\gamma=0,$$

which prove both parts of the proposed question.

For, substituting in the first for pqr their dynamical values, we get between $\alpha\beta\gamma$ the relation

$a^2(b^2-c^2)\,x\,.\,\cos\beta\cos\gamma + b^2(c^2-a^2)\,y\,.\,\cos\gamma\cos\alpha + c^2(a^2-b^2)\,z\,.\,\cos\alpha\cos\beta = 0,$

which is, consequently, the equation of the cone generated by the several lines of impulsive action of the force; and forming, subject to this latter relation, the equation of the envelope of the plane represented by the second for any values of $\alpha\beta\gamma$, we get between $\xi\eta\zeta$ the relation

$$a(b^2-c^2)^{\frac{1}{2}}x^{\frac{1}{2}}\,.\,\xi^{\frac{1}{2}} + b(c^2-a^2)^{\frac{1}{2}}y^{\frac{1}{2}}\,.\,\eta^{\frac{1}{2}} + c(a^2-b^2)^{\frac{1}{2}}z^{\frac{1}{2}}\,.\,\zeta^{\frac{1}{2}} = 0;$$

which is, accordingly, the equation of the cone enveloped by the several axes of initial pure rotation of the body; and therefore, &c., as regards both parts of the question.

Eliminating $\alpha\beta\gamma$ from the three equations of the original groups, we get, for the ruled surface generated by the several axes of rotation, the equation

$$\begin{vmatrix} +(1+b^{-2}\,.\,z\zeta + c^{-2}\,.\,y\eta), & -c^{-2}\,.\,x\eta, & -b^{-2}\,.\,x\zeta \\ -c^{-2}\,.\,y\xi, & +(1+c^{-2}\,.\,x\xi + a^{-2}\,.\,z\zeta), & -a^{-2}\,.\,y\zeta \\ -b^{-2}\,.\,z\xi, & -a^{-2}\,.\,z\eta, & +(1+a^{-2}\,.\,y\eta + b^{-2}\,.\,x\xi) \end{vmatrix} = 0,$$

which surface is, consequently, a cubic scroll, enveloped from the centre of inertia of the body by the quadric cone, the equation of which has been just given.

Note.—If, in the several preceding relations, $\alpha\beta\gamma$ were supposed constant, and xyz instead variable, it would manifestly follow at once from these that, under the same initial circumstances of the body, the several possible lines of action of the single impulsive force, and the several corresponding axes of the initial pure rotation, would generate planes at right angles to each other, whose equations, referred to the three central principal planes of the body, would be $a^2(b^2-c^2)\cos\beta\cos\gamma\,.\,x + \&c. = 0$, and $\cos\alpha\,.\,\xi + \&c. = 0$; properties which were already proved on other principles in No. 4370, page 107, Vol. XXI., of the *Educational Times Reprint* for the year 1874.

6177. (By B. Williamson, M.A.)—Suppose F, F_1, F_2 to be the three foci of a Cartesian oval, F_2 being the exterior focus, and let $FF_1 = c_2$, $FF_2 = c_1$, $F_1F_2 = c$; then, if $mr + lr_1 = nc_2$ be the equation of the inner oval referred to F and F_1, show (1) that its equation referred to F and F_2 is $nr + lr_2 = mc_1$; and, referred to F_1 and F_2, is $mr_2 - nr_1 = lc$. Also prove (2) that the corresponding equations of the outer oval are obtained by changing l into $-l$ in the preceding.

Solution by J. J. Walker, M.A.

This question has been substantially answered by the results of my solution of Prof. Sylvester's Question 4922 (*Reprint*, Vol. XXV., p. 69), wherein, forming the equivalent polar equations with F as pole, and FF_1F_2 as prime radius, it is shown that

$$mr + r_1 = s, \text{ and } \mu r + r_2 = \sigma \ldots\ldots\ldots\ldots\ldots\ldots (1, 2),$$

will represent the same curve, if

$s = \mu c_2(c)$, $\sigma = mc_1(\gamma)$, $(m^2-1)\,c_1 = (\mu^2-1)\,c_2$, or $m^2c_1 - \mu^2c_2 = c_1 - c_2 = c$;

so that, substituting $\frac{m}{l}$ for m, and $\frac{n}{l}$ for μ, (1) and (2) become

$$mr + lr_1 = nc_2, \quad nr + lr_2 = mc_1 \text{ respectively.}$$

By elimination of r between these, in virtue of the relation between cc_1c_2, the third form $mr_2 - nr_1 = lc$ is obtained.

It further appears, that, for a given value of θ, there are two values of r (say r, r'), such that $rr' = c_1c_2(c\gamma)$, one corresponding to $mr + lr_1 = nc_2$, the other to $mr' - lr'_1 = nc_2$; since the elimination of r_1 from the former, and that of r'_1 from the latter, lead to one and the same univectorial polar equation. By eliminating nc_2 between these, $m(r' - r) = l(r'_1 + r)$, which shows that the oval corresponding to the negative value of l is exterior to the other. [See also WILLIAMSON's *Diff. Calc.*, 3rd ed., pp. 411—416.]

5432. (By Professor BALL, LL.D.)—If from any point perpendiculars are drawn to the generators of the surface $z(x^2 + y^2) - 2mxy = 0$; show that the feet of the perpendiculars lie upon a plane ellipse.

Solution by J. J. WALKER, M.A.

The surface is defined as generated by the line

$$\lambda x - y = 0, \quad (1 + \lambda^2)z = 2\lambda m,$$

λ being a variable parameter. The plane through $(x'y'z')$, perpendicular to each of the above planes, is $x + \lambda y = x' + \lambda y'$. It is easily found that the point common to these three planes is

$$(1 + \lambda)^2 x = x' + \lambda y', \quad (1 + \lambda^2)y = \lambda(x' + \lambda y'), \quad (1 + \lambda)^2 z = 2\lambda m;$$

and that these coordinates satisfy the equations

$$x^2 + y^2 = x'x + y'y, \quad 2my'x + 2mx'y - (x'^2 + y'^2)z = 2mx'y',$$

a circular cylinder and plane respectively.

[Another solution is given on p. 96 of Vol. XXX. of the *Reprint*.]

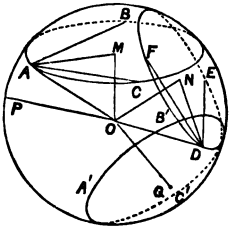
6264. (By R. E. RILEY, B.A.)—If a circle be drawn with its centre at the vertex of the parabola $(x - y)^2 = (32)^{\frac{1}{2}} a(x + y)$, and with a diameter equal to the latus-rectum, prove that the poles with respect to the parabola, of tangents to the circle, lie on the rectangular hyperbola $xy = 2a^2$.

Solution by Prof. SCOTT, M.A.; J. E. A. STEGGALL, B.A.; *and others.*

Turning the axes through 45°, we get, for the parabola, $y^2 = 4ax$; and for the circle, $x^2 + y^2 = 4a^2$; and the polar of (hk) is $yk - 2ak = 2ah$, which touches the circle if $k^2 + 4a^2 = h^2$; changing back our axes, we find that (kh) lies on $xy = 2a^2$.

5507. (By Elizabeth Blackwood.)—If A, B, C, D, E, F are random points on the surface of a sphere, find the probability that the plane through A, B, C will intersect the plane through D, E, F inside the sphere.

Solution by Prof. Seitz, M.A.; Prof. Matz, M.A.; *and others.*



Let M and N be the centres of the sections made by the planes through A, B, C and D, E, F; O the centre of the sphere; OP a radius such that AB is parallel to the plane MOP; OQ a radius such that DE is parallel to the plane NOQ; and A′B′C′ a section of the sphere perpendicular to OQ, and equal to the section ABC. Now, if we give the six points all possible positions, it is evident that the whole number of intersections of the circles ABC and DEF is equal to the whole number of the intersections of the circles A′B′C′ and DEF; hence we will consider those positions of the points in which the circles A′B′C′ and DEF intersect.

Let $OA = r$, $\angle AOM = \theta$, $\angle DON = \phi$, $\angle BAM = \mu$, $\angle CAM = \nu$, $\angle MOP = \lambda$; the angle the plane POM makes with a fixed plane through $PO = \rho$, $\angle EDN = \psi$, $\angle FDN = \omega$, $\angle NOQ = \tau$, and diedral $MOQN = \chi$. Then an element of surface at A is $2\pi r^2 \sin\theta\, d\theta$, at B it is $4r^2 \sin\theta \cos\mu\, d\mu\, d\lambda$, at C it is $4r^2 \sin\theta \sin\lambda \cos\nu \sin(\mu+\nu)\, d\nu\, d\rho$, at D it is $2\pi r^2 \sin\phi\, d\phi$, at E it is $4r^2 \sin\phi \cos\psi\, d\psi\, d\tau$, and at F it is $4r^2 \sin\phi \sin\tau \cos\omega \sin(\psi+\omega)\, d\omega\, d\chi$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi$; of μ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ν, $-\mu$ and $\frac{1}{2}\pi$, and doubled; of λ, 0 and π; of ρ, 0 and 2π; of ψ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ω, $-\psi$ and $\frac{1}{2}\pi$, and doubled; of τ, $\pm(\theta-\phi)$ and $\theta+\phi$ (the double sign being taken + when $\theta > \phi$, and − when $\theta < \frac{1}{2}$); of χ, 0 and 2π.

Hence, since the whole number of ways the six points can be taken, is $(4\pi r^2)^6$, the required chance is

$$p = \frac{4}{(4\pi r^2)^6} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\psi}^{\frac{1}{2}\pi}\int_{\pm(\theta+\phi)}^{\theta+\phi}\int_0^{2\pi} 32\pi r^6 \sin^3\theta\, d\theta \cdot 32\pi r^6$$
$$\times \sin^3\phi\, d\phi \cos\mu\, d\mu \cos\nu \sin(\mu+\nu)\, d\nu \sin\lambda\, d\lambda\, d\rho \cos\psi\, d\psi$$
$$\times \cos\omega \sin(\psi+\omega)\, d\omega \sin\tau\, d\tau\, d\chi$$

$$= \frac{2}{\pi^3} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\psi}^{\frac{1}{2}\pi}\int_{\pm(\theta-\phi)}^{\theta+\phi} \sin^3\theta \sin^3\phi\, d\theta\, d\phi \cos\mu\, d\mu$$
$$\times \cos\nu \sin(\mu+\nu)\, d\nu \sin\lambda\, d\lambda\, d\rho \cos\psi\, d\psi \cos\omega \sin(\psi+\omega)\, d\omega \sin\tau\, d\tau$$

$$= \frac{4}{\pi^3} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\psi}^{\frac{1}{2}\pi} \sin^4\theta \sin^4\phi\, d\theta\, d\phi \cos\mu\, d\mu$$
$$\times \cos\nu \sin(\mu+\nu)\, d\nu \sin\lambda\, d\lambda\, d\rho \cos\psi\, d\psi \cos\omega \sin(\psi+\omega)\, d\omega$$

$$= \frac{1}{\pi^3} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^4\theta \sin^4\phi\, d\theta\, d\phi \cos\mu\, d\mu \cos\nu \sin(\mu+\nu)\, d\nu$$
$$\times \sin\lambda\, d\lambda\, d\rho\, (\pi + 2\psi + 2\cot\psi) \sin\psi \cos\psi\, d\psi$$

$$= \frac{3}{2\pi^2} \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} \sin^4\theta \sin^4\phi\, d\theta\, d\phi \cos\mu\, d\mu \cos\nu \sin(\mu+\nu)\, d\nu$$
$$\times \sin\lambda\, d\lambda\, d\rho$$

$$= \frac{3}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi}\int_0^{\pi} \sin^4\theta \sin^4\phi \, d\theta \, d\phi \cos\mu \, d\mu \cos\nu \sin(\mu+\nu)\, d\nu \times \sin\lambda \, d\lambda$$

$$= \frac{6}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\mu}^{\frac{1}{2}\pi} \sin^4\theta \sin^4\phi \, d\theta \, d\phi \cos\mu \, d\mu \cos\nu \sin(\mu+\nu)\, d\nu$$

$$= \frac{3}{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^4\theta \sin^4\phi \, d\theta \, d\phi \, (\pi + 2\mu + 2\cot\mu) \sin\mu \cos\mu \, d\mu$$

$$= \frac{9}{4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \sin^4\theta \sin^4\phi \, d\theta \, d\phi = \frac{27\pi}{64}\int_0^{\frac{1}{2}\pi} \sin^4\theta \, d\theta = \frac{81\pi^2}{1024}.$$

6192. (By Professor WOLSTENHOLME, M.A.)—Prove that any uniform lamina of mass nM is kinetically equivalent to a system of n equal particles, each of mass M, placed on the ellipse $Ax^2 + By^2 = 2AB$, at points whose successive eccentric angles differ by $\frac{2\pi}{n}$, the axes being the principal axes at the centroid, and nMA, nMB the principal moments of inertia.

Solution by G. J. GRIFFITHS, M.A.; W. J. CONSTABLE, M.A.; *and others.*

Let the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be described on a lamina with its centre coinciding with the centroid, and its axis with the principal axes, of the lamina. Let n particles each of mass m be placed on the ellipse, at points whose eccentric angles differ by $\frac{2\pi}{n}$, θ being the eccentric angle of the first; then the moment of inertia of the particles about the major axis is

$$\mathrm{M}b^2 \cdot \Sigma_{r=0}^{r=n-1} \sin^2\left(\theta + \frac{2r\pi}{n}\right) = \mathrm{M}b^2 \cdot \tfrac{1}{2}n.$$

So the moment of inertia about the minor axis $= \frac{1}{2}n\mathrm{M}a^2$. The product of inertia $\equiv \frac{1}{2}\mathrm{M}ab\Sigma_{r=0}^{r=n-1} \sin 2\left(\theta + \frac{2r\pi}{n}\right)$ vanishes. So that the axes of the ellipse are principal axes for the system of particles as well as for the lamina. Again, since $\Sigma_{r=0}^{r=n-1}\cos\left(\theta + \frac{2r\pi}{n}\right)$ and $\Sigma_{r=0}^{r=n-1}\sin\left(\theta + \frac{2r\pi}{n}\right)$ both vanish, the C. G. of the particles coincides with that of the lamina. Let nM be the mass of the lamina, and nMA, nMB the principal moments of inertia; then the system of particles is kinetically equivalent to the lamina, provided $a^2 = 2\mathrm{B}$, $b^2 = 2\mathrm{A}$, that is, if the points lie on the ellipse

$$Ax^2 + By^2 = 2AB.$$

6234. (By R. RAWSON.)—When the roots of $x^3+nx^2+ax+b=0$ are all real; show, without the aid of CARDAN'S formula, that

$$4bn^3-a^2n^2-18abn+4a^3+27b^2<0,$$

and that one of the roots is less than $\frac{1}{3}\{2(n^2-3a)^{\frac{1}{2}}-n\}$.

Solution by the PROPOSER; A. ANDERSON; *and others.*

If r be a real root of the given cubic (1), the remaining two roots will be determined from the quadratic

$$x^2+(r+n)x+r^2+nr+a=0 \quad \ldots\ldots(2);$$

hence, the roots of (1) are all real when the roots of (2) are real, that is, when $(r+q)^2>4(r^2+nr+a)$, or $\frac{1}{3}\{2(n^2-3a)^{\frac{1}{2}}-n\}>r$.

Since r is a root of (1), we have

$$r^2+nr+a=-\frac{b}{r}, \text{ therefore } r=-\frac{n}{2}\pm\left(\frac{n^2}{4}-a-\frac{b}{r}\right)^{\frac{1}{2}};$$

thus $$\frac{1}{3}\{2(n^2-3a)^{\frac{1}{2}}-n\}>-\frac{n}{2}\pm\left(\frac{n^2}{4}-a-\frac{b}{r}\right)^{\frac{1}{2}};$$

hence $$\left\{\frac{2}{3}(n^2-3a)^{\frac{1}{2}}+\frac{n^2}{6}\right\}^2+a-\frac{n^2}{4}>-\frac{b}{r};$$

therefore $$\left\{\left[\frac{2}{3}(n^2-3a)^{\frac{1}{2}}+\frac{n^2}{6}\right]^2+a-\frac{n^2}{4}\right\}\left[\frac{2(n^2-3a)^{\frac{1}{2}}-n}{3}\right]>-b;$$

which reduces to $$4bn^3-a^2n^2-18abn+4a^3+27b^2<0.$$

If $n=0$, the roots of the cubic $x^3+ax+b=0$ are all real when

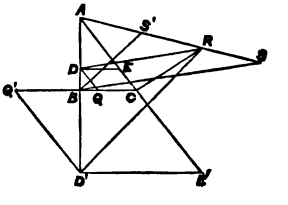
$$4a^3+27b^2<0.$$

[Mr. RAWSON believes the above method of determining the condition for the real roots of the cubic to be "preferable to the usual method of infinite series in extracting the cube root of a binomial (A+iB) as used by Mr. TODHUNTER in his *Theory of Equations*."]

5706. (By the EDITOR.)—Parallel to the base BC of a triangle ABC draw a straight line DE, cutting the sides AB, AC in D, E, such that the squares on BD and CE shall be together equal to the square on DE.

Solution by the PROPOSER.

Draw CR perpendicular to AC and equal to AB; on AR take RS = RS′ = BC; draw RD, RD′ parallel, respectively, to SB, S′B, and meeting AB in D, D′; then the points D, D′ will determine the position of two parallels DE, D′E′ to the base BC, both of which will satisfy the conditions of the problem.



For, draw DQ, D′Q′ parallel to AC;

then we have $CQ^2 : QB^2 = AD^2 : DB^2 = AR^2 : RS^2$
$= AB^2 + AC^2 : BC^2 = BD^2 + DQ^2 : QB^2$;
therefore $BD^2 + DQ^2 = CQ^2$, or $BD^2 + CE^2 = DE^2$.
The same proof obviously applies to D′E′.

If A is a right angle, so that $AB^2 + AC^2 = BC^2$, the points Q, Q′ are the internal and external bisectors of BC, and D, D′ the internal and external bisectors of AB.

6210. (By J. L. McKenzie, M.A.)—Prove that the conics of 5-point contact at any three collinear points on a cubic, meet the curve again in three collinear points.

Solution by L. H. Rosenthal, B.A.; A. L. Selby, M.A.; *and others.*

If of the $3(m+n)$ points of intersection of a curve of the $(m+n)$th order with a cubic, $3m$ lie on a curve of the mth order, the remaining $3n$ lie on a curve of the nth order. Hence, regarding the three conics as a sextic, 15 of the points of intersection with the cubic lie on a curve of the 5th order, viz., on 5 coincident right lines; therefore the remaining 3 intersections are collinear.

6190. (By Professor Townsend, F.R.S.)—A uniform flexible cord, passing without friction through two fixed eyelets in its space, being supposed to hang between them in an arc of an equilateral hyperbola, and beyond them in the two intercepting radii of the curve, its two extremities just reaching the centre; show that the supposed position will be one of free equilibrium under the action of a uniform attractive force emanating from the centre of the curve.

Solution by Prof. Moncourt; J. Hammond, M.A.; *and others.*

The necessary and sufficient conditions for the equilibrium of a string under the action of any central force are

(1) $Tp = \text{const.}$ and $d\mathrm{T} = \mathrm{P}ds \cdot \cos\phi$, or (2) $d\mathrm{T} = \mathrm{P}dr$;

of which (1) expresses the fact that the resultant of the tensions at the extremities of any portion of the string passes through the centre of force; while (2) is the condition that the tangential component of the force $\mathrm{P}ds$, which acts on the element ds of the string, is balanced by the difference of the tensions at its extremities. Now, for the equilibrium of the straight portions of the string, we have, since P is constant $= \mu$, and $\mathrm{T} = 0$ when $r = 0$, the free extremity being at the centre, $\mathrm{T} = \mu r$; and, since there is no friction, this condition also holds along the curve. Substituting in (1), we obtain (3) $pr = \text{const.}$ Now, in the equilateral hyperbola $x^2 - y^2 = a^2$

we have $pr = a^2$, and it is easy to show that, r being variable, the equilateral hyperbola is the only possible form of free equilibrium which satisfies equation (3).

The case of r invariable is a singular solution of the equation formed by eliminating T between (1) and (2), and the circle is a possible form of equilibrium for all laws of force, provided only that the force be repulsive. In fact, since $dr = 0$, (2) gives T = const. for all values of P, and therefore, by (1), p = const., and the constants must be determined so that $p = r$.

[Prof. TOWNSEND remarks that, "in the equilateral hyperbola, regarded as the free catenary of a uniform flexible cord under the action of an attractive force emanating from the centre of the curve, the maintaining force being independent of the distance, and the accompanying tension being that to the centre under its action, throughout the entire extent of any arc of the curve, therefore, &c."]

5968. (By Professor BENJAMIN PEIRCE, LL.D., F.R.S.)—If two bodies revolve about a centre, acted upon by a force proportional to the distance from the centre, and independent of the mass of the attracted body, prove that each will appear to the other to move in a plane, whatever be the mutual attraction.

Solution by C. J. MONRO, M.A.

Let P, m, ρ, P′, m', ρ' be the bodies, their masses, their vectors from C, the centre. Then k, K, K′ being scalars, and K and K′ independent of ρ and ρ', the forces on P and P′ may be written

$$k(\rho'-\rho)-K\rho, \quad k(\rho-\rho')-K'\rho';$$

and the acceleration of P′ relative to that of P will be

$$k\left(\frac{1}{m}+\frac{1}{m'}\right)(\rho-\rho')+\frac{K}{m}\rho-\frac{K'}{m'}\rho',$$

or

$$\left\{k\left(\frac{1}{m}+\frac{1}{m'}\right)+\frac{K}{m}\right\}\rho-\left\{k\left(\frac{1}{m}+\frac{1}{m'}\right)+\frac{K'}{m'}\right\}\rho'.$$

If the force towards C is independent of mass, K′ = K, and the coefficients of the vectors may be in any ratio; therefore, with deference to the distinguished Proposer, I do not see that the relative orbit is under any restriction but that of having CPP′ for instantaneous plane, which it would with any laws of attraction. But if the *acceleration* towards C is independent of mass, or the force *proportional* to mass, $\frac{K'}{m'}=\frac{K}{m}$, and the acceleration, being $\rho-\rho'$ multiplied by a scalar, will be towards P, and the apparent motion will be as if there were no third centre of attraction. The result would be the same if, instead of the "centre," we had a "body" attracting with a force proportional to its mass. But a particular case makes the literal interpretation improbable. If the *force* towards C is independent of mass, we may have P as nearly as we please constantly at rest, by making m great enough, and putting P at rest for once; but thus, as nearly as we please, P′ will simply be a body moving under forces towards two fixed points, with no other limitation than that the force towards one of the points is as the distance. I suppose the proposition will not be true, in general, of this case.

5762. (By Professor SYLVESTER, F.R.S.)—If $f(x, y)$ is a quantic of n dimensions in x, y, prove that $(n-1)\left(\frac{df}{dx}\right)^2 - nf\frac{d^2f}{dx^2}$ is a negative numerical multiplier of y^2 into the Hessian of f.

Solution by W. J. CURRAN SHARP, M.A.

$$H = \begin{vmatrix} u_{11}, & u_{12} \\ u_{12}, & u_{22} \end{vmatrix} = \frac{1}{y}\begin{vmatrix} u_{11}, & yu_{12} \\ u_{12}, & yu_{22} \end{vmatrix} = \frac{n-1}{y}\begin{vmatrix} u_{11}, & u_1 \\ u_{12}, & u_2 \end{vmatrix}$$

[since $xu_{11}+yu_{12} = (n-1)u_1$, and $xu_{12}+yu_{22} = (n-1)u_2$]

$$= \frac{n-1}{y^2}\begin{vmatrix} u_{11}, & u_1 \\ (n-1)u_1, & nu \end{vmatrix} = \frac{n-1}{y^2}\{nuu_{11}-(n-1)u_1^2\},$$ the form required.

A similar reduction of the Hessian of a ternary n-ic leads to the result

$$H = \frac{n-1}{z^2}\begin{vmatrix} u_{11}, & u_{12}, & u_1 \\ u_{12}, & u_{22}, & u_2 \\ (n-1)u_1, & (n-1)u_2, & nu \end{vmatrix}$$

which, being independent of any differentiation with respect to z, is peculiarly applicable when Cartesian coordinates are used.

[The Hessian of the quaternary n-ic admits of a similar reduction, which is applicable when the coordinates are Cartesian, the result being

$$H = \frac{n-1}{w^2}\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_1 \\ u_{12}, & u_{22}, & u_{23}, & u_2 \\ u_{13}, & u_{23}, & u_{33}, & u_3 \\ (n-1)u_1, & (n-1)u_2, & (n-1)u_3, & nu \end{vmatrix}$$

The foregoing solution is more elegant than that given on p. 34 of this volume.]

5753. (By W. R. ROBERTS, M.A.)—Show that the equation of the three tangents at the points of intersection of the straight line and cubic

$$L \equiv \lambda x + \mu y + \nu z, \quad U \equiv x^3 + y^3 + z^3 + 6mxyz,$$

is $$\Sigma U - (1+8m^3)L^2M = 0,$$

where Σ is the condition that L should touch U, and

$$M \equiv [\lambda^4 - 2\lambda(\mu^3+\nu^3) - 6m\mu^2\nu^2]x + [\mu^4 - 2\mu(\lambda^3+\nu^3) - 6m\lambda^2\nu^2]y + [\nu^4 - 2\nu(\lambda^3+\mu^3) - 6m\lambda^2\mu^2]z.$$

[For the value of Σ, see SALMON'S *Higher Plane Curves*, p. 74, 2nd Ed.]

Solution by W. J. CURRAN SHARP, M.A.

If the third evectant of the covariant AL'^2 + &c. (SALMON'S *Higher Plane Curves*, p. 195) be taken, it will be found to be

$$6L^3\{\Sigma U - (1+8m^3)L^2M - 8PLF\},$$

where F is the polar conic of L—so that M is a covariant of the cubic and L—and being of the 4th degree in $\alpha\beta\gamma$, it must be the satellite of L.

Now, $AU + BL^2M = 0$ represents a cubic touching U at the points where it meets L; and if L be a tangent, *i.e.*, if λ, μ, ν satisfy $\Sigma = 0$, the system reduces to L^2M, and therefore $A = \Sigma$, and if the tangents at the points where the curve is cut by z, B will be found $= -(1 + 8m^3)$, and $\Sigma U - (1 + 8m^3) L^2M = 0$ is the equation to the system of tangents. If L be infinity, this is the equation to the asymptotes, and M is the satellite to infinity. By this means the reality of the asymptotes may be determined

[The result may also be obtained by forming the equation

$$(L_1 x + M_1 y + N_1 z)(L_2 x + M_2 y + N_2 z)(L_3 x + M_3 y + N_3 z) = 0,$$

by the help of the binary cubic in SALMON's *Higher Plane Curves*, p. 74.]

NOTE TO THE SOLUTION OF QUEST. 5753. *By* W. J. C. SHARP, M.A.

This interesting question suggests the problem of forming the equation to the n tangents at the points where an n-ic is met by the line $ax + \beta y + \gamma z$. This equation will evidently be the eliminant of the three equations

$$U = 0, \quad L\xi + M\eta + N\zeta = 0, \quad ax + \beta y + \gamma z = 0,$$

and will therefore be of the n^{th} order in $(\xi\eta\zeta)$ the current coordinates, of the $n(n-1)^{th}$ in the quantities $a\beta\gamma$, and of $(2n-1)^{th}$ in the coefficients of U. It will, in fact, be of the form $\Sigma . U + (ax + \beta y + \gamma z)^2 S = 0$, where $\Sigma = 0$ is the tangential equation to the curve, and S the satellite of $ax + \beta y + \gamma z$. For ΣU will be of the proper order in all the quantities, and so will the other term, if S is of the $(2n-1)^{th}$ order in the coefficients, of $(n+1)(n-2)^{th}$ order in $a\beta\gamma$, and of the $(n-2)^{th}$ order in the coordinates, as it should be; while the equation represents an n-ic which touches U at its points of intersection with $ax + \beta y + \gamma z$, and contains that line as a double factor if it be a tangent, and not otherwise. The result of the elimination also gives the equation to the satellite.

6247. (By Professor WOLSTENHOLME, M.A.)—A point O is taken within a triangle ABC, and circles are drawn with centres A, B, C and radii OA sin BOC, OB sin COA, and OC sin AOB respectively; prove that the internal common tangents to these circles, taken two and two, will pass three of them through O, and the other three through another point O′, such that O, O′ are foci of an inscribed ellipse. [If O be taken outside, one pair of internal and two pair of external tangents must be taken.]

I. *Solution by* Professor TOWNSEND, F.R.S.

Denoting by a, b, c the three sides of the triangle ABC, by l, m, n the three lines OA, OB, OC, and by α, β, γ the three angles BOC, COA, AOB respectively; then, the lengths of the three tangents from O to the three

circles being $l\cos\alpha$, $m\cos\beta$, $n\cos\gamma$, and the squares of their three differences in pairs, viz., $(m\cos\beta - n\cos\gamma)^2$, &c., being consequently equal to the squares of the lengths of the internal common tangents to the corresponding pairs of circles, viz., $a^2 - (m\sin\beta + n\sin\gamma)^2$, &c., or

$$m^2 + n^2 - 2mn\cos\alpha - (m\sin\beta + n\sin\gamma)^2, \text{ &c.},$$

or $$m^2 + n^2 - 2mn\cos(\beta+\gamma) - (m\sin\beta + n\sin\gamma)^2, \text{ &c.};$$

therefore &c., as regards the first, and by consequence as regards the second, part of the property in question.

Three circles of any radii being supposed to have their centres at the three vertices of a triangle of any form, it may be easily shown that their three pairs of internal common tangents, or, more generally, any three of their six pairs of common tangents whose intersections are not collinear, are three pairs of tangents to a common conic confocal with that touching the three sides of the triangle at the three corresponding centres of similitude of the three pairs of circles; so that when, as in the question, three of the six tangents pass through a common point, the remaining three also pass through a common point, and the two common points are the two foci of the conic which touches, as aforesaid, the three sides of the triangle, at the three corresponding centres of similitude on its sides.

If a, b, c be the three sides of the triangle, and p, q, r the radii of three opposite circles, it may be easily shown that the equation

$$[a^2-(q+r)^2]^{\frac{1}{2}} + [b^2-(r+p)^2]^{\frac{1}{2}} + [c^2-(p+q)^2]^{\frac{1}{2}} = 0$$

expresses the condition that the conic touched by the six double tangents should break up into a point-pair; for, if u, v, w be the lengths of the three tangents from either point O of the pair, since then, evidently,

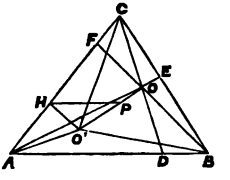
$$u-w = [a^2-(q+r)^2]^{\frac{1}{2}},\ w-u = [b^2-(r+p)^2]^{\frac{1}{2}},\ u-v = [c^2-(p+q)^2]^{\frac{1}{2}};$$

therefore, at once, the relation in question.

II. *Solution by* R. Rawson; Prof. Hall, M.A.; *and others.*

Let CD, AE, BF pass through O; then

$$\sin BCO \cdot \sin CAO \cdot \sin ABO = \sin ACO \cdot \sin BAO \cdot \sin CBO \quad \ldots\ldots(1).$$



Take

$$\angle ACO' = BCO,\ ABO' = CBO,\ BAO' = CAO, \quad \ldots\ldots\ldots\ldots(2);$$

then, by (1), CO', BO', AO' pass through O'.

[The above may be seen neatly demonstrated on p. 260 of Vol. II. of Booth's *New Geometrical Methods.*]

If O, O' be regarded as the foci of an ellipse, then AC, BC will touch it, because $\angle ACO' = BCO$; and BC, AB will touch it, because $\angle CBO = ABO'$; hence the ellipse whose foci are O, O', and semi-axis transverse PH (P being the middle of OO', and O'H perpendicular to AC), is inscribed in the triangle ACB.

Again, $$AO = BO\frac{\sin ABO}{\sin BAO}; \quad \text{and} \quad \sin BOC = \frac{BC}{BO}\cdot\sin BCO;$$

therefore $$AO\cdot\sin BOC = \frac{BC\cdot\sin ABO\cdot\sin BCO}{\sin BAO} \quad \ldots\ldots\ldots\ldots(3),$$

and $$AO'\sin BO'C = \frac{BC\cdot\sin ABO'\cdot\sin BCO'}{\sin BAO'} \quad \ldots\ldots\ldots\ldots(4).$$

From (1) and (2), $AO \cdot \sin BOC = AO' \cdot \sin BO'C$ .....(5);
similarly, $BO \cdot \sin AOC = BO' \cdot \sin AO'C$.....(6),
$CO \cdot \sin AOB = CO' \cdot \sin AO'B$.....(7).

Hence there are two points O, O′ within the triangle ABC by which the same circles can be drawn as described by the question.

From O draw a tangent to the circle whose centre is A, and radius = AQ = AO . sin BOC = AO . sin BOD;

therefore $\sin AOQ = \frac{AQ}{AO} = \sin BOD$, therefore $\angle AOQ = BOD$.

From O draw a tangent to the circle whose centre is B, and radius = BR = BO sin AOC = BO . sin AOD;

therefore $\sin BOR = \frac{BR}{BO} = \sin AOD$, therefore $\angle BOR = AOD$.

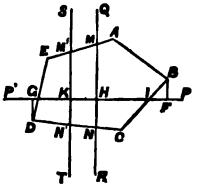
Hence OQ coincides with OR, which is a common tangent to the circles at A and B. Therefore the common internal tangent to the circles at A, B, drawn as in the question, will pass through O. Similarly, the common tangents to the circles at B, C, and A, C will pass through O. The same inferences apply to the point O′.

Similar principles and investigation as the above will apply equally well when the point O is outside the triangle ACB. If O is outside, then O′ is also outside; and, in this case, the points O, O′ may be regarded as the foci of a conic. The position of the lines BO, CO will determine when the conic is an ellipse, hyperbola, or parabola.

5824 & 5903. (By Professor Crofton, F.R.S.) — A convex figure whose perimeter is s and area Δ, is thrown at random on a fixed straight band of breadth a traced on a floor; show that the mean value of the area covered by it is $\frac{\pi a \Delta}{\pi a + s}$ or $\frac{\pi a r^2}{a + 2r}$, when the figure becomes a circle (omitting the cases when the figure does not meet the band).

I. *Solution by* Professor Seitz, M.A.

Let ABCDE be the convex figure, and QRTS the fixed band. Draw PP′ perpendicular to QR, and from B and D, the extreme vertices, draw BF and DG perpendicular to PP′; then GF is the projection of the figure on PP′.



Let HK = a, HF = x_1, KF = x_2, MN = y_1, M′N′ = y_2, GF = z, $\angle$ BIP = θ, area ABCNM = u_1, and area EDN′M′ = u_2.

Then we have $du_1 = y_1 dx_1$, $du_2 = y_2 dx_2$, and for the required mean value

$$M = \int_0^{\pi} \left\{ \int_{-a}^{s} \Delta dx_1 - \int_0^{s} u_1 dx_1 - \int_0^{s} u_2 dx_2 \right\} d\theta + \int_0^{\pi} \int_{-a}^{s} d\theta \, dx_1.$$

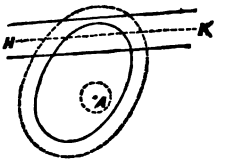
But $$\int_{-a}^{s} \Delta\, dx_1 = (s+a)\,\Delta,$$

and $$\int_0^s u_1 dx_1 + \int_0^s u_2 dx_2 = s\Delta - \int_0^s x_1 y_1 dx_1 + \int_0^s x_2 y_2 dx_2 = s\Delta;$$

therefore $\mathrm{M} = \int_0^\pi a\Delta\, d\theta + \int_0^\pi (s+a)\, d\theta$. By the solution of Quest. 5582 (*Reprint*, Vol. XXX., pp. 50, 51), we have $\int_0^\pi s\, d\theta = s$; $\therefore \mathrm{M} = \frac{\pi a \Delta}{\pi a + s}$.

For the circle, we have $s = 2\pi r$, $\Delta = \pi r^2$, and $\mathrm{M} = \frac{\pi a r^2}{a + 2r}$.

II. *Solution by the* PROPOSER.



Let M (B) be the mean value of the portion of the band covered by the figure; the chance that a point, taken at random on the figure, shall fall on the band is $p = \frac{\mathrm{M}(\mathrm{B})}{\Delta}$; which is the same as if the figure were fixed, and the band thrown at random upon it. Now fix the random point at A: then the favourable cases are when HK, the bisector of the band, meets a circle, centre A, radius $\frac{1}{2}a$; and the whole number of cases are when HK meets a curve *parallel to* the given contour S, and at a distance $\frac{1}{2}a$ from it. As the perimeter of this curve is $\mathrm{S} + 2\pi(\frac{1}{2}a)$, we have $p = \frac{2\pi(\frac{1}{2}a)}{\mathrm{S} + 2\pi \cdot \frac{1}{2}a}$; being independent of the position of A; therefore $\mathrm{M}(\mathrm{B}) = \frac{\pi a \Delta}{\pi a + \mathrm{S}}$.

4031. (By Professor SYLVESTER, F.R.S.)—Define *a tree* in its most general form regarded as a complex of relations between any given number of points or ideas.

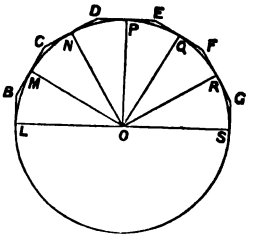
Solution by the PROPOSER.

(1) A ramification is a system of relations between points such that every point is in relation with some one or more other points, but so that no system of successively related points forms a circuit returning into itself.

(2) A tree is an irreducible ramification, that is, one which is not an aggregate of two or more ramifications.

Such is the *negative definition* of a tree, but we have also a positive *intuition* of a tree, or, say, of arborescence, and from the combination of the positive conception with the negative condition is hatched an illimitable series of hidden consequences constituting the science of the subject. Such is, I believe, the general story of the birth and growth of all pure science; so it is that the *egg is laid*.

5357. (By Professor SYLVESTER, F.R.S.) In the annexed figure, O is the centre of the circle; LB, BC, CD, DE... are any number of tangents to the circle at L, M, N, P, ...; BC is bisected at M, CD at N, DE at P, ..., so that 2LB = BC = CD ... = 2GS. The lines touching the circle, and the radii drawn to the points of contact, are now supposed to be jointed rods moveable freely round each other at either extremity, and it is easily seen that the fan-like figure thus formed becomes a complete linkage, one (that is to say) capable of changing its form in only a single definite series of ways.



1. Prove that, when the fan is opened or closed, the alternate angles LOM, NOP, QOR, ..., always remain equal to each other, as also the alternate angles MON, POQ, ROS, ..., so that the fan may be used to divide any angle into a given number of parts,—as, for example, in the figure above, when the fan is open or closed, the angle LON will always remain one-third of the angle LOS.

2. Imagine planes to become rigidly attached to BC, CD, DE, ..., and prove that, when OL is fixed, the trajectories of points taken in them are unicursal curves of the respective orders.

Solution by W. J. CURRAN SHARP, M.A.

Any figure CMON is such that CMO = CNO in every position; therefore DNO = OMB; and OD bisects NOP, OC bisects MON, and OB bisects LOM; and BOM = DON = FOQ; therefore LOM = NOP = &c., and MON = POQ = &c., and therefore $LON = \frac{1}{n} LOS$.

This construction derives a trisecting machine from the trisection of a right angle, and in the same way a machine for dividing an angle into n parts may be derived from the division of the right angle into n parts.

6198. (By Dr. HOPKINSON, F.R.S.)—At the Paris Exhibition a freezing machine (Giffard's) was exhibited, consisting of two cylinders, the pistons of which work on to two cranks on the same shaft, driven by an external source of power, and of a large air reservoir which is always maintained at the temperature of the external air. In the first cylinder air is compressed till its pressure is the same as in the reservoir, when valves open and the air passes, as the stroke is completed, into the reservoir. The second and smaller cylinder acts as an engine receiving compressed air from the reservoir for such a portion of the stroke that, being expanded for the remainder of the stroke, it is discharged at atmospheric pressure, but at a lower temperature. If V_1 and V_2 be the volumes of the cylinders, and Π the atmospheric pressure, and if the compression and expansion be supposed adiabatic, prove that the work done during each stroke in the first and second cylinders is

$$\Pi V_1 \frac{\gamma}{\gamma-1} \cdot \frac{V_1-V_2}{V_2}, \quad \Pi \frac{\sim}{\gamma-1} (V_1-V_2).$$

Solution by J. J. WALKER, M.A.

Let V'_1, V'_2 be the volumes of the parts of the two cylinders occupied by air after compression has ceased and before expansion has commenced, respectively, the pressure in each case being P, as in the reservoir. It is supposed that the mass of air forced into the reservoir from the larger, is equal to that admitted into the smaller cylinder; so that (1) $PV'_2 = \Pi V_1$. The efficient work (W_2) done in V_2 is $W'_2 + W''_2 - W'''_2$; W'_2 being the whole work done by the air admitted before expansion begins: viz.,

$$W'_2 = PV'_2 = \Pi V_1;$$

W''_2 the whole work done during expansion of this air: viz.,

$$W''_2 = \int_{V'_2}^{V_2} p dV = c \int_{V'_2}^{V_2} V^{-\gamma} dV = \frac{c}{\gamma - 1} (V^{1-\gamma+1} - V_2^{-\gamma+1})$$
$$= \Pi (\gamma - 1)^{-1} (V_1 - V_2),$$

since $c = PV_2'^{\gamma} = \Pi V_2^{\gamma}$ and $PV'_2 = \Pi V_1$; lastly, W'''_2 being the work spent in overcoming the resistance of atmospheric pressure: viz. $W'''_2 = \Pi V_2$. Hence $W_2 = \gamma (\gamma - 1)^{-1} \Pi (V_1 - V_2)$.

The work (W_1) done by engine-power in the larger cylinder V_1 may be deduced from W_2 by observing that the limiting pressures and mass of gas are the same in both cases; or its value may be obtained independently thus: $W_1 = W''_1 + W'_1 - W'''_1$, W''_1 being the whole work done in compressing the air: viz.,

$$W''_1 = \int_{V'_1}^{V_1} p dV = c' \int_{V'_1}^{V_1} V^{-\gamma} dV = (\gamma - 1)^{-1} (PV'_1 - \Pi V_1),$$

since $c' = PV_1'^{\gamma} = \Pi V_1^{\gamma}$; W'_1 being the whole work done in forcing the compressed air into the reservoir: viz., $W'_1 = PV'_1$; lastly, W'''_1 being the part of the whole work $W''_1 + W'_1$ done by atmospheric pressure: viz., $W'''_1 = \Pi V_1$. Now, $PV_1'^{\gamma} = \Pi V_1^{\gamma}$ and $PV_2'^{\gamma} = \Pi V_2^{\gamma}$; whence $V'_1 = V_1 V'_2 \div V_2$ and $PV'_1 = V_1 PV'_2 \div V_2 = \Pi V_1^2 \div V_2$, by (1). Thus

$$W_1 = \gamma (\gamma - 1)^{-1} (PV'_1 - \Pi V_1) = \gamma (\gamma - 1)^{-1} \Pi V_1 (V_1 - V_2) \div V_2.$$

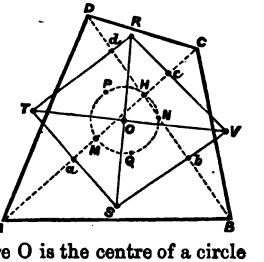
6297. (By C. TAYLOR, M.A.)—The tangent at P to a rectangular hyperbola meets a chord OQ at right angles in R; show that the diameters bisecting the chords PO and PQ bisect the angles between the diameters to P and R.

Solution by Prof. GENESE, M.A.; H. T. GERRANS, B.A.; *and others.*

Let M be the middle point of PO. Then CP, CM are conjugate in direction to RP, MP respectively. Thus angle PCM = angle RPM. But a circle on PO as diameter would pass through R. Therefore MP = MR, and angle MPR = angle MRP. Thus angle PCM = angle MRP, and MPCR lie on a circle. But MP = MR. Therefore MC bisects angle PCR. Again, $RP^2 = RO \cdot RQ$. Therefore QPO is a right angle. The diameters bisecting PO and PQ must therefore be at right angles.

6282. (By Prof. Genese, M.A.)—ABCD is a quadrilateral, M, N are the middle points of the diagonals AHC, BHD; and the circles HAB, HCD meet in P, and the circles HBC, HDA in Q; prove that the points M, N, H, P, Q lie on one circle.

Solution by F. D. Thomson, M.A.; Prof. Scott, M.A.; *and others.*



Let R, S, T, V be the centres of the circles round HCD, HAB, HAD, HBC: then RTSV is a parallelogram whose sides bisect HA, HB, HC, HD at right angles in the points a, b, c, d.

Let O be the intersection of RS, TV. Then $ac = \frac{1}{2}$ AC = MC, therefore aM = cH; but Oc = Oa, therefore OH = OM. Similarly OH = ON; therefore O is the centre of the circle round HMN. But, since HP is the common chord of two circles, centres R and S, HP is bisected at right angles by RS, therefore OH = OP. Similarly OH = OQ, therefore O is the centre of a circle through HNQMP.

6318. (By J. Venn, M.A.)—A, B, C are set to sort a heap of books; A taking the English political books, and the bound foreign novels; B the bound political works and the English novels, provided they are not political; and C the bound English works, and the unbound political novels. What works will be claimed by both A and B? Will any be claimed by all three?

I. *Solution by* C. J. Monro, M.A.; R. R. Grey, M.A.; *and others.*

In an obvious notation, say that A, B, C, take

$$ep + be'n,\ bp + enp',\ be + b'np.$$

Here ep and bp have bep in common, leaving $b'ep$ (which has nothing in common with enp') and $be'p$ (which has only $be'np$ in common with $be'n$). So A's and B's lots have in common only $bep + be'np$, or bound political books which are either English or novels, including bound *Coningsbys*. It is now plain that C's lot has only bep, bound English political books, in common with A's and B's.

II. *Solution by* H. McColl, B.A.

Take one book from the heap, and speaking of this book let α, β, γ respectively denote the statements, A *claims it*, B *claims it*, C *claims it*; and let e, p, b, n respectively denote the statements, It is *English*, It is *political*, It is *bound*, It is a *novel*.

Thus our data are $\alpha = ep + be'n$, $\beta = bp + enp'$, $\gamma = be + b'pn$; and the inference is evident that

$$\alpha\beta = ebp + bpe'n = bp\,(e + e'n) = bp\,(e + n), \quad \alpha\beta\gamma = bep + bepn = bep.$$

That is to say, we infer that both A and B claim every *bound political* work, provided it be *English* or a *novel* (or both); and that A, B, C all three claim the *bound English political* works. The symbol $=$ is here used throughout, for all the implications are true *conversely*. The last equation, for example, $\alpha\beta\gamma = bep$, is equivalent to the double implication

$$(\alpha\beta\gamma : bep)\,(bep : \alpha\beta\gamma).$$

III. *Solution by the* PROPOSER.

Let a, b, c stand respectively for the assignments to the three persons; x, y, z, w for the books which are English, political, bound, and novels (*foreign* being equivalent to *not*-English). Then, in BOOLE's and my notation, the data stand thus:—

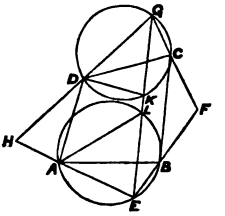
$$a = xy + z\bar{x}w, \quad b = yz + \bar{x}yw, \quad c = xz + y\bar{z}w.$$

Now, community of class membership is determined by (logical) multiplication. Therefore the assignment common to A and B, namely ab, is $xyz + \bar{x}yzw$; and that to A, B, and C, namely abc, is xyz; since $x\bar{x} = 0$, $y\bar{y} = 0$, &c.; which may be read off in the same words as Mr. MCCOLL's answer.

6336. (By Rev. H. G. DAY, M.A.)—Through four given points draw lines which will form a quadrilateral similar to a given quadrilateral.

Solution by Dr. LIEBER; J. O'REGAN; H. MURPHY; *and others.*

Let A, B, C, D be the given points, and EFGH the quadrilateral required, the form of which is given. Since we know CD and $\angle$ CGD, the circle round $\triangle$CDG is also known, which may cut EG in K; $\angle$ CDK being $=$ $\angle$ CGK, and the latter being known, the point K is determined. Besides the circle round $\triangle$ AEB is known which cuts EG in L; and $\angle$ LAB $=$ $\angle$ LEB; consequently L is fixed and the line KL found, which cuts the circles in E and G.



6296. (By E. B. ELLIOTT, M.A.)—A circular city has n gates, n straight streets along radii to these gates, and n circular streets concentric with its wall cutting these at right angles. A person starts from the

centre along any one of the streets and turns whenever possible, but never inwards towards the centre. Show (1) that the whole number of possible routes by which he may thus leave the city is $n2^n$; and that the chance that the gate by which he leaves the city is the one in the direction towards which he originally started is, (2), if n is odd, 2^{1-n}; or (3), if n is even, $2^{1-n} + \frac{n!}{2^n (\frac{1}{2}n!)^2}$.

Solution by W. A. WHITWORTH, M.A.; J. HAMMOND, M.A.; *and others.*

1. He may start in n directions, and after that there are n occasions on which he has choice of two routes; therefore the total number of ways is $n2^n$.

2. If n be odd, the only means by which he can leave off in the direction in which he began, is by turning always to the right or always to the left; that is, there are (for each direction in which he may start) two ways; and therefore the chance is 2^{1-n}.

3. If n be even, the same end may be gained, not only as in the last case, but also by turning $\frac{1}{2}n$ times to the right and $\frac{1}{2}n$ times to the left; that is, there are $2 + \frac{n!}{(\frac{1}{2}n!)^2}$ ways; and therefore the chance is that given in the Question.

5948. (By A. MARTIN.)—A person throws 10 dice, and those that fall "ace up" are taken away. He throws the remainder, and those that fall "ace up" are taken away, and so on, till every die has been turned "ace up." Required the probability that this will be effected in 4 throws.

Solution by W. J. C. SHARP, M.A.; Prof. MATZ, M.A.; *and others.*

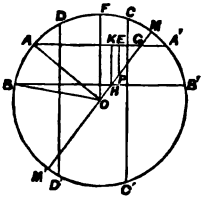
The chance is the same as that of throwing an ace with each die at least once in four throws. Now the chance with each die is

$$\frac{1}{6} + \frac{5}{36} + \frac{25}{216} + \frac{125}{1296} = 1 - (\tfrac{5}{6})^4;$$

hence the chance with all the dice $= [1 - (\frac{5}{6})^4]^{10}$ = the chance required, or, more generally, turning ace up with every one of m dice in n throws $= [1 - (\frac{5}{6})^n]^m$.

5833. (By COLONEL CLARKE, C.B., F.R.S.)—Two pairs of parallel lines cross a circle so as to form a rectangle within it; prove that (1) the chance that this rectangle contains a given point at the distance δ from the centre of the circle is $\frac{1}{4}(1 - \delta^2)^2$; and (2) the chance that it contains a portion of a given diameter of the circle is $\frac{1}{4}(1 + \log_e 4)$.

Solution by Prof. SEITZ; Prof. CASEY; *and others.*



1. Let AA′, BB′ and CC′, DD′ be the two pairs of parallel lines, P the given point, and O the centre of the circle. Draw OF and PE parallel to CC′. Let OA = 1, OP = δ, $\angle$ POF = θ, $\angle$ AOF = ϕ, $\angle$ BOF = ψ, $\phi_1 = \sin^{-1}(\delta \sin\theta)$, and $\phi_2 = \cos^{-1}(\delta\cos\theta)$. The number of ways the two pairs of parallel lines can cross the circle so as to form a rectangle within it, AA′ being farther from the centre than BB′, is

$$2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_\phi^{\pi-\phi} AA'^2\,d\theta\sin\phi\,d\phi\sin\psi\,d\psi = 16\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin^3\phi\cos\phi\,d\theta\,d\phi = 2\pi.$$

The number of ways the rectangle can be formed, so as to contain the given point, is

$$\int_0^{\frac{1}{2}\pi}\Big\{\int_{\phi_1}^{\phi_2}\int_{\phi_2}^{\pi-\phi} 2AE\,.\,A'E\sin\phi\,d\phi\sin\psi\,d\psi + \int_{\pi-\phi_2}^{\pi-\phi_1}\int_{\pi-\phi}^{\phi_2} 2AE\,.\,A'E\sin\phi\,d\phi\sin\psi\,d\psi\Big\}d\theta$$

$$= 2\int_0^{\frac{1}{2}\pi}\Big\{\int_{\phi_1}^{\phi_2}(\sin^2\phi-\delta^2\sin^2\theta)(\cos\phi+\delta\cos\theta)\sin\phi\,d\phi - \int_{\pi-\phi_2}^{\pi-\phi_1}(\sin^2\phi-\delta^2\sin^2\theta)(\cos\phi+\delta\cos\theta)\sin\phi\,d\phi\Big\}d\theta$$

$$= \int_0^{\frac{1}{2}\pi}(1-\delta^2)^2\,d\theta = \tfrac{1}{2}\pi(1-\delta^2)^2.$$

Hence the chance that the rectangle contains the given point is

$$\tfrac{1}{2}\pi(1-\delta^2)^2 + 2\pi = \tfrac{1}{2}(1-\delta^2)^2.$$

2. Let MM′ be the given diameter intersecting AA′, BB′ in G, H. Draw HK parallel to CC′. Let $\angle$ MOF = θ, $\angle$ AOF = ϕ, $\angle$ BOF = ψ, and $\psi_1 = \cos^{-1}(\sin\phi\cot\theta)$. The number of ways the rectangle can be formed so as to contain a portion of the given diameter is

$$2\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\Big\{\int_{\psi_1}^{\pi-\psi_1}(2AK\,.\,A'K+A'K^2)\sin\psi\,d\psi + \int_{\pi-\psi_1}^{\pi-\phi} AA'^2\sin\psi\,d\psi\Big\}d\theta\sin\phi\,d\phi$$

$$+ 2\int_0^{\frac{1}{2}\pi}\int_\theta^{\frac{1}{2}\pi}\int_\phi^{\pi-\phi}(2AG\,.\,A'K-KG^2)\,d\theta\sin\phi\,d\phi\sin\psi\,d\psi$$

$$= \tfrac{8}{3}\int_0^{\frac{1}{2}\pi}\Big\{\int_0^{\theta}(3\cos\phi+\sin\phi\cot\theta)\sin^3\phi\,d\phi + \int_\theta^{\frac{1}{2}\pi}(3\sin^2\phi+3\sin\phi\cos\phi\tan\theta-2\cos^2\phi\tan^2\theta)\sin\phi\cos\phi\,d\phi\Big\}d\theta$$

$$= \int_0^{\frac{1}{2}\pi}\big\{1+(\tfrac{1}{2}\pi-\theta)\tan\theta+\theta\cot\theta\big\}\,d\theta = \int_0^{\frac{1}{2}\pi}(1+2\theta\cot\theta)\,d\theta$$

$$= \tfrac{1}{2}\pi - 2\int_0^{\frac{1}{2}\pi}\log\sin\theta\,d\theta = \tfrac{1}{2}\pi(1+\log_e 4).$$

Hence the chance that the rectangle contains a portion of the given diameter is $\tfrac{1}{2}\pi(1+\log_e 4)\div 2\pi = \tfrac{1}{4}(1+\log_e 4)$.

6291. (By R. Tucker, M.A.)—A certain hyperbola is the locus of the mid-points of chords of an ellipse, the perpendicular bisectors of which pass through a fixed point; find the envelop of this hyperbola when the fixed point lies on the auxiliary circle of the ellipse.

Solution by R. Knowles, B.A., L.C.P.; J. E. Steggall, B.A.; *and others.*

If the equation of the ellipse be $b^2x^2 + a^2y^2 = a^2b^2$, and (X, Y) the given point, the equation of the hyperbola is obviously $\frac{a^2(X-x)}{x} = \frac{b^2(Y-y)}{y}$; and if $X^2 + Y^2 = a^2$, we may put $X = a\cos\theta$, $Y = a\sin\theta$, and the equation is $\frac{a^2\cos\theta}{x} - \frac{b^2\sin\theta}{y} = \frac{a^2-b^2}{a}$, and the envelop is

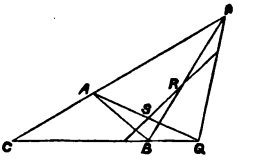
$$\frac{a^4}{x^2} + \frac{b^4}{y^2} = \left(\frac{a^2-b^2}{a}\right)^2,$$

the reciprocal polar of the evolute of the ellipse with respect to the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = a$.

6303. (By D. Edwardes.)—If a quadrilateral, three of whose sides are fixed, be inscribable in a circle, prove that the line bisecting its diagonals touches a fixed parabola, with respect to which the fixed triangle is self-conjugate, and whose directrix passes through one of the vertices of the triangle.

Solution by Rev. T. W. Openshaw, M.A.; Prof. Scott, M.A.; *and others.*

Let ABQP be the quadrilateral and PQ the variable side; then $\angle BQP = \pi - BAP = CAB$ = constant; so also APQ is constant; therefore PQ moves parallel to itself, and ABC is a fixed triangle.



Let $CQ = q$, $CP = p$;
mid-points of PB, $\frac{1}{2}a$, $\frac{1}{2}p$; of AQ, $\frac{1}{2}q$, $\frac{1}{2}b$;
then the equation to RS is

$$x(p-b) + y(q-a) + \tfrac{1}{2}(ab-pq) = 0;$$

but $p = mq$, therefore $\frac{1}{2}mq^2 - (mx+y)q + bx + ay - \frac{1}{2}ab = 0$;
therefore the envelope is $(mx+y)^2 = m(2bx + 2ay - ab)$;
the polar of (0, 0) is $bx + ay - ab = 0$, *i.e.* AB;
the polar of $(a, 0)$ is $2mxa + may = m\{b(x+a) + ay - ab\}$, or $x = 0$, *i.e.* CA.

5909. (By the Editor.)—A rod, capable of turning on a given point in its length, is placed at a fixed distance from a centre of force that varies inversely as the square of the distance; find its oscillations.

I. *Solution by* J. J. Walker, M.A.

Let A be the centre of force, μ its intensity on unit mass at unit distance; C the lower end of the rod BC, oscillating, about the point O, in a given plane through OA. Let a be the cross section, ρ the density, and a, a' the segments OC, OB of the rod; θ the angle it makes with OA at the time t; AD, AE the perpendicular on BC and the line bisecting the angle BAC respectively; b the length of AO. Then the small angles OAC, OAB are equal to $a\theta+(b-a)$, $a'\theta+(b+a')$ respectively; whence

$$\angle \mathrm{BAC} = (a+a')\,b\theta+(b-a)\,(b+a'),$$

and
$$\angle \mathrm{OAE} = \left\{(a-a')\,b+2aa'\right\}\theta+2\,(b-a)\,(b+a').$$

Now the resultant attraction on the mass of the rod is along AE, and is equal to $\mu\rho a\,\mathrm{BAC}+\mathrm{AD}$, and its moment about O is $\mu\rho a\,\mathrm{BAC}\,.\,\mathrm{OAE}+\theta$. Hence the angular acceleration is

$$\ddot{\theta}=3\mu(a+a')\,b\left\{(a-a')\,b+2aa'\right\}\theta+2(a^3+a'^3)(b-a)^2(b+a')^2, \text{ say, } \mu k\theta+k';$$

and the length of the equivalent simple pendulum is $l=gk'+\mu k$. In order that the movement should be stable, $a : a' > \mathrm{AC} : \mathrm{AB}$.

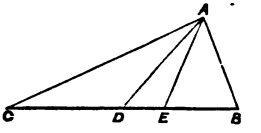
In the case of O being the centre, or a half the length, of BC, $k' : k = (b^2-a^2)^2 : 3b$.

In the case of the point O coinciding with the end B, or a being the whole length of BC, $k' : k = 2a\,(b-a)^2 : 3$; which, when A is very distant, and $\mu+b^2=g$, gives $(a+b$ vanishing$)$ $2a+3$ as the length of the simple pendulum equivalent to a rod of length a oscillating about one end—as it should.

II. *Solution by* G. S. Carr, B.A.

Let A be the centre of force, BC $= 2b$ the rod, D its centre, AE the bisector of $\angle$BAC, AD $= a$, $\angle$BDA $= \theta$.

The resultant attraction of A upon BC acts along EA, and is $= 2\mu \sin \mathrm{BAD} + a \sin\theta$ (Todhunter's *Statics*, Art. 204). The moment of this force about D is



$$2\mu \sin \mathrm{BAD} \sin \mathrm{DAE} + \sin\theta$$
$$= \mu\,(\cos \mathrm{DAC} - \cos \mathrm{DAB}) + \sin\theta.$$

Let $x = \tan \mathrm{DAB} = \dfrac{b \sin\theta}{a-b\cos\theta}$, $y = \tan \mathrm{DAC} = \dfrac{b\sin\theta}{a+b\cos\theta}$,

then the above becomes $\mu\left\{(1+y^2)^{-\frac{1}{2}}-(1+x^2)^{-\frac{1}{2}}\right\}+\sin\theta$.

Expand this, then we may write the equation of motion as follows:—

$$k^2\frac{d^2\theta}{dt^2}+\frac{\mu}{\sin\theta}\left\{\tfrac{1}{2}(x^2-y^2)-\tfrac{3}{8}(x^4-y^4)+\&c.\right\}=0.$$

For small oscillations about the position of equilibrium, for which $\theta=0$, we may neglect the terms after (x^2-y^2); hence, by putting $k^2=\frac{1}{2}b^2$, $x=\dfrac{b\theta}{a-b}$, $y=\dfrac{b\theta}{a+b}$, we obtain $\dfrac{d^2\theta}{dt^2}+\dfrac{6\mu ab}{(a^2-b^2)^2}\theta=0$;

and the time of oscillation $= 2\pi\,(a^2-b^2)\div(6\mu ab)^{\frac{1}{2}}$.

If a be very great, $t = 2\pi\,(a^2-2ab)^{\frac{1}{2}}\div(6\mu b)^{\frac{1}{2}}$,

increasing with the distance.

6302. (By J. Young, B.A.)—If AB, AD be the greater sides of a quadrilateral ABCD inscribed in a circle, prove that

$$\frac{\sin(A+B)}{\sin(A+D)} = \frac{AB^2-CD^2}{AD^2-BC^2}.$$

Solution by G. Eastwood, M.A.; E. B. Elliott, M.A.; *and others.*

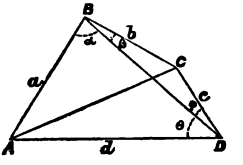
In the annexed diagram, we have

$\sin(A+B) = \sin(\pi-\alpha-\theta+\alpha+\phi) = \sin(\theta-\phi)$;

similarly $\sin(A+D) = \sin(\alpha-\beta)$;

also $\sin(\theta+\phi) = \sin(\alpha+\beta)$;

$$\therefore \frac{\sin(A+B)}{\sin(A+D)} = \frac{\sin^2\theta-\sin^2\phi}{\sin^2\alpha-\sin^2\beta} = \frac{AB^2-CD^2}{AD^2-BC^2}.$$



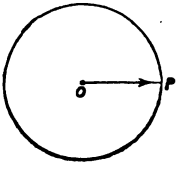
6299. (By C. Leudesdorf, M.A.)—A circular hoop of radius a, lying on a smooth horizontal plane, is rotating with angular velocity ω about an axis through its centre perpendicular to its plane, and a particle is projected from the centre of the hoop with velocity v against its inner surface, supposed perfectly rough; show that, if e be the coefficient of elasticity, and n the ratio of the mass of the particle to that of the hoop, the particle will strike the hoop again after a time

$$\frac{2(2n+1)^2 aev}{(2n+1)^2e^2v^2+(n+1)^2a^2\omega^2}.$$

Solution by J. E. A. Steggall, B.A.; D. Edwardes; *and others.*

Let P be the particle at the instant of striking. To find the velocity along OP after the blow, we have, if bodies were inelastic, $nv=(n+1)\,V$, $nV = \frac{nv}{n+1}$; but bodies are elastic; and therefore, after the blow, the velocities of hoop and particle are

$$\frac{nv}{n+1}(1+e), \quad \frac{nv}{n+1}-\frac{ev}{n+1} \quad \text{.....(1).}$$



Also, Ω being the angular rotation of the hoop after the blow, U the velocity of the centre perpendicular to OP, and W the velocity of the particle, we have $W = U + a\Omega$ (through perfect roughness),

$a\Omega + nW = a\omega$ (by moments), and $U + nW = 0$ (no external forces);

therefore $$W = \frac{a\omega}{2n+1}, \quad U = -\frac{na\omega}{2n+1} \quad \text{.....(2);}$$

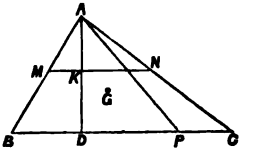
therefore relative velocity of particle in this direction $= \frac{a\omega(n+1)}{2n+1}$; and by (1), in direction OP, $= -ev$.

Now $$\text{time} = \frac{\text{chord}}{\text{resultant vel.}} = \frac{2a\cos\theta}{\text{result. vel.}}$$
$$= \frac{2a(\text{vel. along PO})}{(\text{result. vel.})^2} = \frac{2aev(2n+1)^2}{e^2v^2(2n+1)^2 + (n+1)^2a^2\omega^2}.$$

6275. (By Prof. Cropton, F.R.S.)—Show that the mean distance of the vertex of a triangle from all points in the area is equal to its distance from the centre of gravity, *measured along a parabolic path*, which leaves the vertex in the direction of one of the sides, and reaches the centre of gravity in a direction parallel to the other,—the axis of the parabola being parallel to the base.

I. *Solution by* G. Heppel, M.A.; Prof. Matz, M.A.; *and others.*

Take A as origin, AD as positive axis of y. Let AD $= h$, DC $= l$, DB $= m$, DP $= z$, and let G be the centre of gravity. Let BC $= l+m = a$, then the mean distance of A from BC $= \frac{1}{a}\int_{-m}^{l} (h^2+z^2)^{\frac{1}{2}}\,dz = p$. And, if AK $= y$, mean distance from MN $= \frac{yp}{h}$;



therefore, if P $=$ mean distance from all points in the triangle,

$$P = \int_0^h \frac{yp}{h}\cdot\frac{ya}{h}\cdot\frac{2}{ah}\,dy = \tfrac{2}{3}p = \frac{2}{3a}\int_{-m}^{l}(h^2+z^2)^{\frac{1}{2}}\,dz,$$ or, putting $z^2 = 3ax$,

we have $$P = \int_{\frac{m^2}{3a}}^{\frac{l^2}{3a}} \left(\frac{h^2}{3ax}+1\right)^{\frac{1}{2}} dx.$$

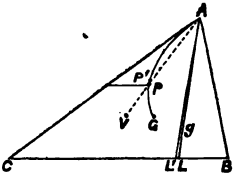
But the length of an arc of any parabola $(y-b)^2 = 4d(x-c)$, taken between the limits $x=e$ and $x=f$, is $\int_f^e \left(\frac{d+x-c}{x-c}\right)^{\frac{1}{2}} dx$; hence P $=$ length of such an arc, the latus rectum being $\frac{4h^2}{3a}$, and the limits of integration $e = \frac{l^2}{3a}+c$ and $f = \frac{m^2}{3a}+c$. Let $f=0$, then $c = -\frac{m^2}{3a}$ and $e = \frac{1}{3}(l-m)$, and the parabola is $(y-b)^2 = \frac{4h^2}{3a}\cdot\left(x+\frac{m^2}{3a}\right)$.

Take $b = \frac{2hm}{3a}$, and the equation becomes $y^2 - \frac{4hmy}{3a} = \frac{4h^2x}{3a}$, which passes through the vertex where $x = f = 0$. If $x = e = \frac{1}{3}(l-m)$, we get $y = \frac{2}{3}h$; therefore the parabola passes through the centre of gravity.

For intersection with AB, put $x = -mh^{-1}y$, and we get $y^2 = 0$; therefore AB is a tangent.

The equation to a parallel to AC through G is $y - \frac{1}{3}h = \frac{h}{l}\{x - \frac{1}{3}(l-m)\}$; and, for the intersection of this with the parabola, we get $(y - \frac{1}{3}h)^2 = 0$; therefore this parallel is also a tangent.

II. *Solution by the* PROPOSER.



Conceive a line AL to turn continuously round A, from the position AB to AC; and, as it revolves, imagine that the whole mass of the triangle *to the right* of AL is collected into the vertex A. The centre of gravity will thus move from G to A, along a certain curve. As the line moves from the position AL to the consecutive one AL′, the centre moves from the position P to P′ on its locus; PP′ being parallel to AL; also $PP' = \frac{ALL'}{ABC} \cdot \frac{2}{3}AL$, as the mass ALL′ has been transferred to A, its centre of gravity g having moved over gA.

Hence, if $ABC = \Delta$, we have arc $AG = \frac{1}{\Delta} \Sigma (ALL' \times \frac{2}{3}AL)$ for all positions of AL. Now, to find the mean distance of A from all points in the triangle, we have to multiply every element of the area by its distance from A, and divide by Δ; but the portion of this sum due to elements on the triangle ALL′ is $ALL' \times \frac{2}{3}AL$, therefore mean distance = arc AG.

It is easy to see that the curve AG is a parabola, as when the centre of the whole mass is at P, it must be in a line with A and the centre V of the triangle ACL; so that, if A be the origin, and the axes be AC and a parallel to CB, $2x : y = CL : b$; also $AP : AV = CL : a$; whence $CL = \frac{3ay}{2b}$; therefore $y^2 = \frac{4b^2}{3a}x$.

The mean distance of a given point from any plane area is equal to the length of the path of the centre of gravity of the mass, supposed continuously transferred to the given point as above.

"AT RANDOM" AGAIN. BY C. J. MONRO, M.A.

The proper meaning of distribution "at random" has been largely discussed in various volumes of the *Reprints*, but perhaps not exhaustively. Questions of science have not, I venture to think, been sufficiently discriminated from questions of the convenient use of language: the difference has not always been observed in practice, and I do not find that at any stage of the controversy it has been laid down in theory. It appears to me

that the phrase covers two quite different things; one a strictly scientific notion, the other a more or less appropriate convention.

It is universally agreed that if a point is taken at random in space it is equally likely to be in any two portions of equal volume. What is the justification of this? Simply the fact that (in Euclidean space of course, and with a limitation to be pointed out below) it is impossible to lay down any other condition which is not either self contradictory, as that different probabilities should belong to equal volumes of different form, or insufficiently general, as that different probabilities should belong to equal volumes at different distances from St. Paul's. This is an absolutely definite mathematical notion, and it is probably what they have in view who say that, things taken at random are taken "according to no law" (*Reprint*, VI., 82, VII., 83, 84): but as I do not pretend to attach a definition to a received term, I shall use another word, *indifference*, and say that a locus is placed or given *indifferently* when it is impossible to indicate consistently where or how it is placed except by reference to things not implied in such determination as it has. Cases of indifference are those of a point in space, in an infinite plane, on a sphere, on a right cylinder, on an infinite straight line, on the circumference of a circle, on a helix. So of an infinite straight line or plane in space, or an infinite straight line in an infinite plane. If a straight line or plane passes through a given point, its disposition is indifferent: given the disposition, the direction or aspect is indifferent. A sphere or circle of given size is indifferent in other respects. A locus of given size or shape is indifferent as to the place of any one of its points and the directions of any two not parallel straight lines in it. But two points cannot be taken indifferently, because in their determination as "two points" is implied a mutual distance; and with reference to this as greater or less, it is possible to indicate consistently how the points are placed; we may suppose them more likely to be nearer or further apart. It is true there is no natural unit of distance in the Euclidean world; but there is a natural distinction between directions of change in distance; increase is absolutely different from decrease. In general, a locus cannot be indifferent in size or shape. A point cannot be taken indifferently within a bounded range, because it may be more likely to be nearer or further from the boundary; and in connexion with certain problems it may be interesting to observe a limitation alluded to above, that loci cannot be taken indifferently in an infinite space which is conceived as other than spherical or circular, because there may be different probabilities in different assignable directions. An abstract quantity cannot be given indifferently, because it, or the real quantities in it, may be more likely to be nearer or further from naught: but time may, because we know of no natural zero-point of time.

Without suggesting that the term "at random" should be restricted to this meaning, I maintain that when it has not this meaning it has no scientific meaning; and, in any particular case in which this meaning is inadmissible, you cannot say that such and such a meaning is right or wrong, but only that it is more or less convenient. There are indeed simple and general cases in which the convenience is great, and the convention made is easy and obvious. Thus, although two points cannot be taken indifferently, we may agree that a second shall be considered as taken independently of the first. So when a point is said to be taken at random within a given range, it is of course easily understood that the taking differs from that made in unbounded space (as an infinite plane or the whole surface of a sphere) only by the exclusion of exterior points. And the case of abstract magnitude may be likened to that of concrete measurement from merely

arbitrary zeros. But more convention is required if we are to speak of taking at random the size or shape of a locus, and perhaps the intention ought to be specified in each case. It is unreasonable to select some conspicuous constant or constants, as the radius of a circle or the axes of an ellipse, and suppose uniform variation of the same. Why not the square of a selected constant (which indeed one can find a reason for, in the case of a circle in a plane, but only such a reason as would require the cube of the radius of a sphere), and why not any selected function of it? If any general convention is to be made, there is one which seems less arbitrary than any other, namely, to suppose such variations equally probable as affect uniformly the probability that a point taken indifferently will be enclosed by the locus in question; that is, to suppose uniform variation of the content, the length, area, or volume.* If, in course of such variation, successive boundaries intersect, the whole of the differential content between them must of course nevertheless be taken with the same sign: it will therefore be seen that this convention would give a condition for variation in mere position, and the same condition as indifferent distribution would give. The same convention may be applied to the distribution of lines or surfaces within ranges of space, as of chords in a circle. If I was obliged to give a meaning to the term "random chord," I should prefer the chord which cut off a uniformly varying area, so that $\pi^{-1}(\phi - \sin\phi)$ would be the probability that it subtended at the centre an angle less than ϕ. But I submit that the term is too arbitrary, and that the discussions about it in the *Reprints* are a good illustration of the misuse of the word *random*. Several meanings have been assigned, different from that just suggested. One, depending on uniform distribution of the extremities, has been recommended or rather laid down in language that would not want emphasis if applied to the doctrine of the conservation of energy, but without any scientific justification that I can find. The reason offered by Mr. Woolhouse (*Reprint*, V, 110, Solution of Question 1894) does not appear to me to meet the only difficulty, namely, why the distribution of the chords is to be governed by the distribution of their extremities. Miss Blackwood's argument (XXIX, 19) that we do in fact define a chord by reference to its extremities, is precisely to the point, and that is much; but I venture to think the unanswerable answer had been given already by Mr. Godfray (VII, 66), that this definition is not the only possible one. The utmost that can be said is that it is the usual one, the most ancient one it may be,† or the most elegant; but to decide on such grounds is not science, but literary criticism.

Closely connected with random distribution is the difficult subject (discussed in XXIII, 68—70; XXXI, 100—103); if a bag contains a given number of balls about which nothing is known except that they are black and white, whether "every possible partition of the number" or "every possible partition of the *things* (treated as 'different')" is *a priori* equally likely. I call it a difficult subject, because any full discussion of it goes to the root of the mathematical theory of probability, and I would ask any person who is quite content with the latter assumption, what he thinks

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* This convention would give at once without calculation the answer "an even chance" to the Editor's Question 6016.

† Is it, though? It seems implied by the metaphor contained in the word. But I find neither the word nor the metaphor in Euclid. It may be said however that ἐνταθῆναι, used in Plato's Menon, 86, 87, of inscribing a triangle of given area, expresses the same metaphor.

of the following bit of dialogue:—"You are absolutely ignorant how these things are divided between the two categories; I must not ask you to tell me even approximately the probability that they are divided about equally." "Yes, *certainty*, if there are enough of them." But whatever the difficulties of the subject, I cannot think there is any difficulty in answering the bare question: the answer is, neither the one nor the other; the probability of any specified partition is indeterminate. The case is not one of indifference, as is by hypothesis the question whether the white or the black are the more numerous; therefore the equal probability of the possible partitions is on both interpretations neither axiomatic nor demonstrable. The point is not as new as at least one writer seems to think: BOOLE raised it in the 20th chapter of his *Laws of Thought*. In fact, he connected it with the subject of this paper, but in a way in which, as the reader will easily understand, I do not see how to follow him.

6161. (By Prof. GENESE, M.A.)—If a straight line through the focus of a conic meet the tangents at the ends of the transverse axis in P, Q, prove that the circle whose diameter is PQ will touch the conic.

Solution by W. E. WRIGHT, B.A.; G. TURRIFF, M.A.; *and others*.

Let the two straight lines be $x^2 = a^2$, and the focus $(c, 0)$, and let the straight line make an angle θ with the axis; then the equation of the circle on PQ as diameters will be

$$(x^2-a^2)+[y+(c-a)\tan\theta][y+(a+c)\tan\theta]=0,$$

or
$$x^2+y^2-a^2+2cy\tan\theta+(c^2-a^2)\tan^2\theta=0,$$

and the envelop is $c^2y^2+(a^2-c^2)(x^2+y^2-a^2)=0$, or $\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$,

a conic to which the straight lines $x^2 = a^2$ are the tangents at the ends of the transverse axis, and the point $(c, 0)$ is a focus. The conic is an ellipse if $a > c$, and an hyperbola if $a < c$. For real contact, the line $cy+(c^2-a^2)\tan\theta=0$ must meet the conic in real points,

therefore
$$\frac{a^2-c^2}{c^2}\tan^2\theta<1, \quad \tan^2\theta<\frac{c^2}{a^2-c^2},$$

which is always true in the hyperbola; but in the ellipse the line must lie between the two straight lines joining the focus to the ends of the minor axis.

5982. (By S. TEBAY, B.A.)—Find three numbers such that, if the cube of their sum be increased by the product of every two of them, the three sums shall be cubes.

Solution by the Rev. Dr. KNISELY; H. POLLEXFEN, B.A.; *and others.*

Let ax, ay, az be the numbers, and ax_1, ay_1, az_1 the roots of the respective cubes; then, if we suppose $x+y+z=1$, we shall have

$$a^3+a^2xy=a^3x_1^3,\ a^3+a^2x_1z_1=a^3y_1^3,\ a^3+a^2y_1z_1=a^3z_1^3;$$

and, putting $x_1^3-1=p$, $y_1^3-1=q$, $z_1^3-1=r$, and $Q=pq+pr+qr$, we find $a=\frac{xy}{p}=\frac{xz}{q}=\frac{yz}{r}$; $x=\frac{pq}{Q}$, $y=\frac{pr}{Q}$, $z=\frac{qr}{Q}$; therefore $a=\frac{pqr}{Q^2}$; and the three numbers and the roots of the three cubes are, respectively,

$$\frac{p^2q^2r}{Q^3},\ \frac{p^2qr^2}{Q^3},\ \frac{pq^2r^2}{Q^3};\ \frac{pqr}{Q^2}x_1,\ \frac{pqr}{Q^2}y_1,\ \frac{pqr}{Q^2}z_1.$$

[In *numbers*, one correspondent finds $\frac{63\,.\,234}{323^3}$, $\frac{26\,.\,234}{323^3}$, $\frac{234^2}{323^3}$.]

5914. (By R. TUCKER, M.A.)—How must a ball be struck from the edge of a circular billiard table, so that, after n reflexions, it may come back to the point of departure? Discuss the question for other positions of the ball.

Solution by W. J. C. SHARP, M.A.

If a_1, a_2, &c. be the successive angles made by the path with the radii at the points of impact, $\tan a_1=\epsilon\tan a_2=\epsilon^2\tan a_3$, &c. .....(1). Now, in the first case, $a_1+a_2+\ldots+a_n=\frac{1}{2}(n-2)\pi$, therefore $\tan(\Sigma a)=0$ or ∞ according as n is even or odd. And, by substituting their values from (1) for $\tan a_2$, $\tan a_3$, &c., an $(n-1)$-ic equation is obtained to determine a_1. If the point be not on the circumference, but at a distance $a<r$ from the centre, and if

$$\sin\theta=\frac{r}{r-a}\sin a_1,\ \text{and}\ \sin\phi=\frac{r}{r-a}\sin a_n\ \ldots\ldots\ldots\ldots(2),$$

$$a_1+2a_2+2a_3+\ldots 2a_{n-1}+a_n+\theta+\phi=(n-2)\pi,$$

therefore $$\tan(a_1+2a_2+2a_3+\ldots+a_n+\theta+\phi)=0,$$

from which, by (1) and (2), an equation to determine $\tan a_1$ can be deduced.

6147. (By E. W. SYMONS, B.A.)—If A, B, C be the angles of a spheric triangle, and x, y, z any real quantities, prove that

$$x^2+y^2+z^2-2yz\cos A-2zx\cos B-2xy\cos C\ (=\Pi\ \text{say})\ \text{is positive.}$$

Solution by Prof. WOLSTENHOLME; W. WILKINS, B.A.; *and others.*

$$\Pi\equiv(x-y\cos C-z\cos B)^2+\{y\sin C-z\,\mathrm{cosec}\,C\ (\cos A+\cos B\cos C)\}^2$$
$$+z^2\,\mathrm{cosec}^2\,C\{\sin^2B\sin^2C-(\cos A+\cos B\cos C)\}^2;$$

and hence (for real values of x, y, z) Π is always positive, if

$$\sin^2 B \sin^2 C - (\cos A + \cos B \cos C)^2$$

is positive, or if $[\cos A + \cos (B-C)]\,[-\cos A - \cos (B+C)]$ be positive, *i.e.*, if $\cos S + \cos (S-A) \cos (S-B) \cos (S-C)$ be negative, where $2S = A+B+C$, which is always true when A, B, C are angles of a spherical triangle, S being $> \frac{1}{2}\pi$, and $S-A$, $S-B$, $S-C < \frac{1}{2}\pi$.

Otherwise: $\Pi \equiv X^2 + \ldots + 2yz \cos (\pi - A) + \ldots \equiv$ square of distance from origin of (x, y, z) referred to oblique axes inclined at angles $\pi-A$, $\pi-B$, $\pi-C$, and is therefore always positive. [The question is an extension of Problem 513, in the 2nd edition of Prof. WOLSTENHOLME's *Problems*.]

6300. (By J. L. MCKENZIE, B.A.)—In any triangle that is not equilateral, if Δ be the area, and l, m, n be the lines bisecting the angles and terminated by the opposite sides, prove that

$$lmn > \frac{64\Delta^3}{(b+c)(c+a)(a+b)}.$$

Solution by H. MURPHY; J. O'REGAN; *and others.*

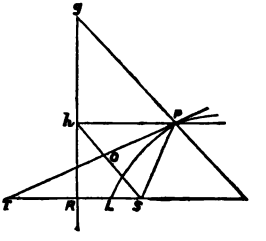
$2\Delta = (b+c)\, l \sin \frac{1}{2}A = (c+a) \sin \frac{1}{2}B = (a+b) \sin \frac{1}{2}C$; therefore

$$lmn = \frac{8\Delta^3}{(b+c)(c+a)(a+b) \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C} > \frac{64\Delta^3}{(b+c)(c+a)(a+b)},$$

since (WOLSTENHOLME's *Problems*, Ex. 517) $\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C < \frac{1}{8}$.

6304. (By J. HAMMOND, M.A.)—If at any point of a parabola the normal be produced to meet the directrix in g, prove that the radius of curvature at P is double the length of Pg.

Solution by C. BICKERDIKE, L.C.P.; W. E. WRIGHT, B.A.; *and others.*



$Pg : P\lambda = P\lambda : O\lambda\,(SO)$,

therefore $SO \cdot Pg = P\lambda^2 = PS^2$.

But $R = \frac{2SP^2}{SO} = \frac{2SO \cdot Pg}{SO}$

$= 2Pg$.

6258. (By H. McColl, B.A.)—Assuming as an established law that the event C happens whenever A happens with X, and also whenever B happens without X; that C never happens when B happens with X, nor when A happens without X; that A happens on an average 3 times out of 5, and B once out of 3 times; that A, when it happens, is followed by C once in 10 times (on an average), and B by C 11 times in 20. Show that X will happen on an average between 63 and 83 times in 300.

Solution by the Proposer.

Definitions.—The symbol x_a denotes the chance that the statement x is true *on the assumption that a is true.* The symbol ϵ is an equivalent for the symbol 1, and denotes any statement whose truth is assumed *throughout the whole of an investigation.* Hence, the symbol x simply denotes *the chance that x is true.*

The data of the question are $a_\epsilon = \frac{3}{5}$, $b_\epsilon = \frac{1}{3}$, $c_a = \frac{1}{10}$, $c_b = \frac{11}{20}$, with the two implications $ax + bx' : c$ and $bx + ax' : c'$; and we are required to find x_ϵ, or its limits.

Solving the given implications with respect to x and x', we get

$$bc' + ac : x, \text{ and } ac' + bc : x'.$$

Hence $(bc' + ac)_\epsilon$ is an inferior limit to x_ϵ, and $1-(ac' + bc)_\epsilon$ is a superior limit.

Now, $(bc' + ac)_\epsilon = (bc')_\epsilon + (ac)_\epsilon = b_\epsilon c'_b + a_\epsilon c_a = \frac{3}{20} + \frac{3}{50} = \frac{63}{300}$,

and $1-(ac' + bc)_\epsilon = 1 - a_\epsilon c'_a - b_\epsilon c_b = 1 - \frac{27}{50} - \frac{11}{60} = \frac{83}{300}$.

Thus the required chance is between $\frac{63}{300}$ and $\frac{83}{300}$.

6031. (By the Editor.)—What is the simplest possible six-valued function of six symbols 123456 ?

Solution by the Rev. T. P. Kirkman, M.A., F.R.S.

The six values of the 6-valued function made with 123456 are ($11 = 1^2$)

$$\begin{aligned}
&1^2 2^2 34 + 5^2 1^2 42 + 6^2 4^2 31 + 2^2 4^2 15 + 5^2 2^2 46 \\
&+ 5^2 4^2 61 + 6^2 2^2 35 + 2^2 1^2 56 + 6^2 3^2 51 + 1^2 3^2 64 \\
&+ 3^2 1^2 25 + 3^2 6^2 24 + 5^2 3^2 26 + 2^2 6^2 14 + 5^2 6^2 21 \\
&+ 6^2 5^2 43 + 1^2 4^2 53 + 1^2 6^2 45 + 5^2 4^2 32 + 3^2 4^2 12 \\
&+ 2^2 3^2 16 + 5^2 2^2 13 + 4^2 1^2 62 + 3\ 5^2 41 + 5^2 1^2 63 \\
&+ 4^2 6^2 52 + 4^2 3^2 65 + 3^2 2^2 54 + 6^2 1^2 23 + 4^2 2^2 36 = A_1,
\end{aligned}$$

which is the simplest possible algebraic function of six letters that has six values. The five values are got by putting 1 for 2, and 2 for 1, 13 for 31, 14 for 41, 15 for 51, 16 for 61; thus,

$$2^2 1^2 34 + 5^2 2^2 41, \&c. = A_2,\ 3^2 2^2 14 + 5^2 3^2 42, \&c. = A_3,\ 4^2 2^2 31 + 5^2 4^2 12, \&c. = A_4,$$
$$5^2 2^2 34 + 1^2 5^2 42, \&c. = A_5,\quad 6^2 2^2 34 + 5^2 6^2 42, \&c. = A_6;$$

and no substitution—*e.g.*, 216534 or 625341—will give a new value, but merely change the order of terms in some value. Of course the first terms of A_1 and A_2 are the same.

5113. (By the Editor.)—124, 154, 156, 163, 234, 132, 473, 456, 467, 367, and 124, 143, 126, 234, 167, 523, 173, 562, 567, 573 are two sets of ten triplets each made with seven elements, so that every duad employed occurs twice in either system. If we change them by any substitution, as 2135746—which means, put 2 for 1, 1 for 2, 3 for 3, 5 for 4, 7 for 5, 4 for 6, and 6 for 7—the first becomes 215, 275, 274, 243, 135, 231, 563, 574, 546, 346, which is similar to the first; but no substitution can make the first into the second, or the second into the first; so that these two are dissimilar sets. Find the entire number of dissimilar sets.

Solution by the Rev. T. P. Kirkman, M.A., F.R.S.

The five sets are as follows:—

124, 154, 156, 163, 234, 132, 473, 456, 467, 367;
124, 143, 126, 234, 167, 523, 173, 562, 567, 573;
126, 267, 743, 732, 451, 745, 567, 165, 123, 341;
123, 236, 674, 275, 762, 574, 541, 134, 346, 152;
254, 432, 123, 341, 265, 541, 675, 571, 162, 167;

which are the five Δ-faced 10-edra, of four different symmetries, the figures denoting the summits. My method shows that no more are possible. You may construct any one on any triangle, writing four figures inside it.

6209. (By C. Leudesdorf, M.A.)—A uniform inextensible string AB of length a is lying in a smooth horizontal plane in the form of the spiral whose intrinsic equation is $\log \frac{s}{c} = \frac{3\phi}{2}$, A being the point where $\phi = 0$. It is acted on by an impulsive tension at A; find the impulsive tension at any point P of the string; and show that, if P begin to move in a direction making an angle θ with the tangent at P to the spiral, then

$$\text{arc AP} = a\left(\frac{\tan\theta + 2}{2 - 4\tan\theta}\right).$$

Solution by Prof. Wolstenholme, M.A.; Prof. Genese, M.A.; *and others.*

The intrinsic equation of an equiangular spiral being $s = ce^{\phi\cot\alpha}$, let T, T + δT be the impulsive tensions at P, Q, arc to P = s, arc PQ = δs, u, v the tangential and normal velocities instantaneously communicated to PQ, $\kappa\delta s$ the mass of PQ; then we have

$$\kappa\delta s \,.\, u = (\mathrm{T} + \delta\mathrm{T})\cos\delta\phi - \mathrm{T}, \text{ or } \kappa u = \frac{d\mathrm{T}}{ds},$$

$$\kappa\delta s \,.\, v = (\mathrm{T} + \delta\mathrm{T})\sin\delta\phi, \qquad \text{or } \kappa v = \mathrm{T}\frac{d\phi}{ds},$$

and, since PQ is unchanged in length,

$$\frac{du}{ds} = v\frac{d\phi}{ds}, \text{ therefore } \frac{d^2\mathrm{T}}{ds^2} = \mathrm{T}\left(\frac{d\phi}{ds}\right)^2;$$

and, eliminating s by the given equation, $\frac{d^2T}{d\phi^2} - \frac{dT}{d\phi}\cot\alpha = T$; whence $T = A_1\epsilon^{m_1\phi} + A_2\epsilon^{m_2\phi}$, where m_1, m_2 are roots of the equation $z^2 - z\cot\alpha = 1$,

and
$$\tan\theta = \frac{v}{u} = \frac{T\,d\phi}{dT} = \frac{A_1\epsilon^{m_1\phi} + A_2\epsilon^{m_2\phi}}{m_1 A_1\epsilon^{m_1\phi} + m_2 A_2\epsilon^{m_2\phi}},$$
$$A_1\epsilon^{m_1\phi}(m_1\tan\theta - 1) = A_2\epsilon^{m_2\phi}(1 - m_2\tan\theta);$$

therefore
$$\frac{A_1}{A_2}\left(\frac{s}{c}\right)^{\frac{m_1-m_2}{\cot\alpha}} = \frac{1-m_2\tan Q}{m_1\tan\theta - 1};$$

and, since the tension at the free end is 0, $\frac{A_1}{A_2}\left(\frac{a}{c}\right)^{(m_1-m_2)\tan\alpha} = -1$,

or $\left(\frac{s}{a}\right)^{(m_1-m_2)\tan\alpha} = \frac{m_2\tan\theta - 1}{m_1\tan\theta - 1}$, *i.e.*, arc AP $= a\left(\frac{m_2\tan\theta - 1}{m_1\tan\theta - 1}\right)^{\frac{\cot\alpha}{m_1-m_2}}$;

and when $\cot\alpha = \frac{3}{4}$, $m_1 = 2$, $m_2 = -\frac{1}{2}$, giving arc AP $= a\left(\frac{\tan\theta + 2}{2 - 4\tan\theta}\right)$.

Thus the general result is arc AP $= a\left(\frac{m + \tan\theta}{m - m^2\tan\theta}\right)^{\frac{m^2-1}{m^2+1}}$,

where $m - m^{-1} = \cot\alpha$; or where the intrinsic equation of the spiral is

$$\log\frac{s}{c} = (m - m^{-1})\cot\phi;$$

and a is the length of the arc measured from the pole (not from the point where $\phi = 0$, but where $\phi = -\infty$). Also, s (or AP) is measured from the pole of the spiral, since, by the given equation, $s = 0$ when $\phi = -\infty$.

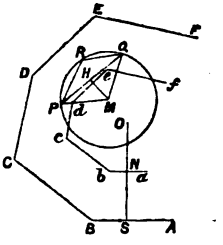
1588. (By Professor Sylvester, F.R.S.)—Solve the equation
$$ax^5 + 5bx^4 + 10cax^3 + 10cbx^2 + 5c^2ax + c^2b = 0.$$

Solution by the Proposer.

Let $x + c^{\frac{1}{2}} = u$, $x - c^{\frac{1}{2}} = v$, $A + B = a$, $(A - B) = bc^{-\frac{1}{2}}$; then $Au^5 + Bv^5 = 0$. Hence $\frac{u}{v}$, and consequently x, becomes known.

6223. (By Professor Seitz, M.A.)—Within the perimeter ($= s$) of an n-gon circumscriptible about a circle of radius r, three points are taken at random; prove that (1) the chance that the circle through these three points will lie wholly within the polygon is $\frac{1}{8}\pi^2 r^2 s^{-2}$; and (2) when the polygon becomes a circle, the chance (which is then that of the Editor's Question 1843) is $\frac{1}{8}$.

Solution by the PROPOSER.



Let ABCDEF ... be the polygon; PQR the triangle formed by joining the three random points P, Q, R; O the centre of the inscribed circle of the polygon; and M the centre of the circle circumscribing PQR. Draw the polygon *abcdef* ..., making its sides parallel to those of ABCDEF..., and at a distance from them equal to MP; and draw ONS and MH perpendicular to AB and PQ.

Now, while PQ is given in length and direction, and the angle PRQ is given, if MP is less than OS, the area of the polygon *abcdef* ... represents the number of ways the three points can be taken, so that the circle circumscribing the triangle will lie wholly within the given polygon.

Let PQ $= 2x$, OS $= r$, perimeter of ABCDEF ... $= s$, area of segment PRQ $= t$, area of sector PMQ $= v$, area of triangle PMQ $= u$, area of polygon ABCDEF ... $= \Delta$, $\angle$PMP $= \theta$, $\phi = \sin^{-1}(x \div r)$, and $=$ the angle which PQ makes with some fixed line.

Then we have PM $= x \operatorname{cosec}\theta$, ON $= r - x\operatorname{cosec}\theta$,

$$\text{area } abcdef = \frac{\Delta}{r^2}(r - x\operatorname{cosec}\theta)^2,\ v = \theta x^2 \operatorname{cosec}^2\theta,\ u = x^2\cot\theta,$$

$$t = v - u, \text{ and } dt = dv - du = 2x^2\operatorname{cosec}^2\theta\,(1 - \theta\cot\theta)\,d\theta.$$

An element of the polygon at Q is $4x\,dx\,d\psi$, or $4r^2\sin\phi\cos\phi\,d\phi\,d\psi$, and at R it is dt; the limits of x are 0 and r, and of ϕ, 0 and $\frac{1}{2}\pi$; those of θ, ϕ and $\pi - \phi$; and of ψ, 0 and 2π. Hence, doubling, since R may lie on either side of PQ, we have for the required chance

$$p = \frac{2}{\Delta^2}\int_0^r\int_{\theta=\phi}^{\theta=\pi-\phi}\int_0^{2\pi}\frac{\Delta}{r^2}(r - x\operatorname{cosec}\theta)^2 .\ 4x\,dx\,dt\,d\psi$$

$$= \frac{16\pi}{r^2\Delta^2}\int_0^r\int_{\theta=\phi}^{\theta=\pi-\phi}(r - x\operatorname{cosec}\theta)^2\,x\,dx\,dt$$

$$= \frac{32\pi}{r^2\Delta^2}\int_0^r\int_{\phi}^{\pi-\phi}(r - x\operatorname{cosec}\theta)^2\,(1 - \theta\cot\theta)\,x^3\,dx\operatorname{cosec}^2\theta\,d\theta$$

$$= \frac{2\pi r^4}{3\Delta^2}\int_0^{\frac{1}{2}\pi}\{2(\pi - 2\phi)\sin 2\phi + 8\sin^2 2\phi - 3\sin^2 2\phi\cos 2\phi + 64\sin^4\phi\cos\phi\log\tan\tfrac{1}{4}\phi\}\,d\phi$$

$$= \frac{2\pi^2 r^4}{5\Delta^2} = \frac{8\pi^2 r^2}{5s^2}.$$

When the polygon becomes a circle, $s = 2\pi r$, and we have $p = \frac{2}{5}$.

[See *Reprint*, Vol. VIII., p. 90; XIII., 17, 95; XVI., 50.]

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